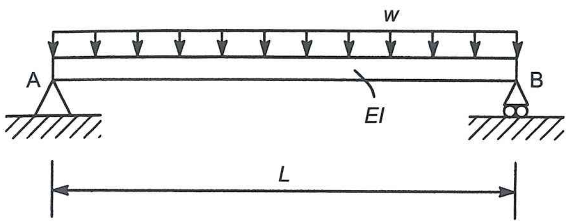


FIGURE P.15.8

P.15.9 Calculate the deflection of the free end C of the cantilever beam ABC shown in Fig. P.15.9 using the unit load method.

Ans. $wa^3(4L-a)/24EI$. (downwards)

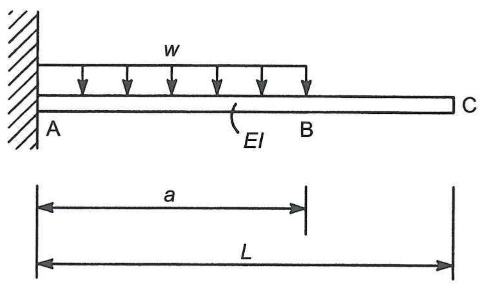


FIGURE P.15.9

P.15.10 Use the unit load method to calculate the deflection at the free end of the cantilever beam ABC shown in Fig. P.15.10.

Ans. $3WL^3/8EI$. (downwards)

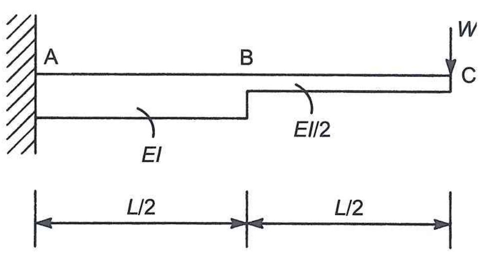


FIGURE P.15.10

P.15.11 Use the unit load method to find the magnitude and direction of the deflection of the joint C in the truss shown in Fig. P.15.11. All members have a cross-sectional area of 500 mm^2 and a Young's modulus of 200 000 N/mm^2.

Ans. 23.4 mm, 9.8° to left of vertical.

P.15.12 Calculate the magnitude and direction of the deflection of the joint A in the truss shown in Fig. P.15.12. The cross-sectional area of the compression members is 1000 mm^2 while that of the tension members is 750 mm^2. Young's modulus is 200 000 N/mm^2.

Ans. 30.3 mm, 10.5° to right of vertical.

P.15.13 A rigid triangular plate is suspended from a horizontal plane by three vertical wires attached to its corners. The wires are each 1 mm diameter, 1440 mm long with a modulus of elasticity of

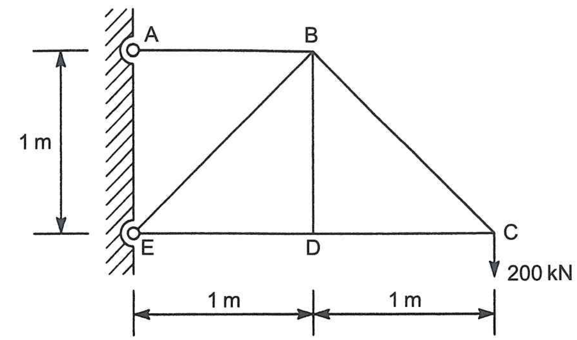


FIGURE P.15.11

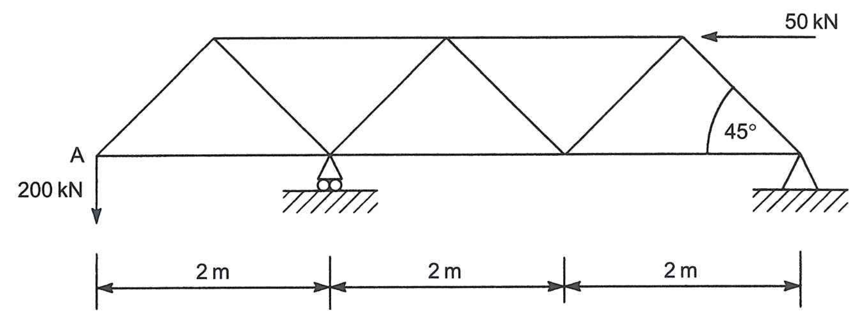


FIGURE P.15.12

deflection at the point of application of a load of 100 N placed at a point equidistant from the three sides of the plate.

Ans. 0.33 mm.

P.15.14 The pin-jointed space truss shown in Fig. P.15.14 is pinned to supports 0, 4, 5 and 9 and is loaded by a force P in the x direction and a force $3P$ in the negative y direction at the point 7.

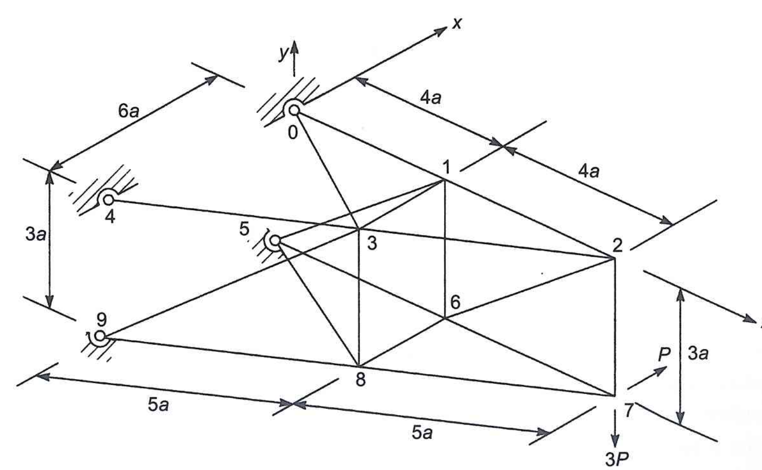


FIGURE P.15.14

Find the rotation of the member 27 about the z axis due to this loading. All members have the same cross-sectional area, A, and Young's modulus, E. (Hint. Calculate the deflections in the x direction of joints 2 and 7.)

Ans. $382P/9AE$.

P.15.15 The tubular steel post shown in Fig. P.15.15 carries a load of 250 N at the free end C. The outside diameter of the tube is 100 mm and its wall thickness is 3 mm. If the modulus of elasticity of the steel is 206 000 N/mm^2, calculate the horizontal movement of C.

Ans. 53.3 mm.

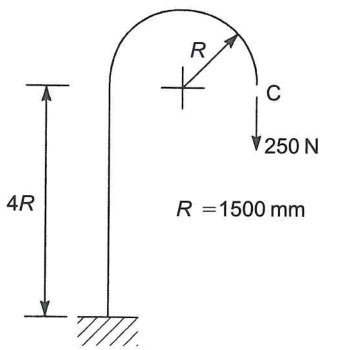


FIGURE P.15.15

P.15.16 Figure P.15.16 shows a cantilever beam subjected to linearly varying temperature gradients along its length and through its depth. Calculate the deflection at the free end of the beam.

Ans. $2\alpha t_o L^2/3h$.

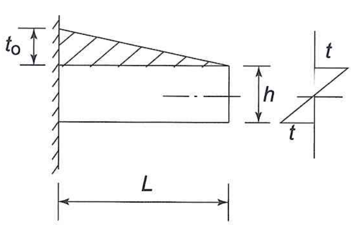


FIGURE P.15.16

P.15.17 A cantilever beam of length L and depth h is subjected to a uniform temperature rise along its length. At any section, however, the temperature increases linearly from t_1 on the undersurface of the beam to t_2 on its upper surface. If the coefficient of linear expansion of the material of the beam is α, calculate the deflection at its free end.

Ans. $\alpha(t_2-t_1)L^2/2h$.

P.15.18 A simply supported beam of span L is subjected to a temperature gradient which increases linearly from zero at the left-hand support to t_0 at the right-hand support. If the temperature gradient also varies linearly through the depth, h, of the beam and is zero on its undersurface, calculate the deflection of the beam at its mid-span point. The coefficient of linear expansion of the material of the beam is α.

Ans. $-\alpha t_0 L^2/48h$.

P.15.19 Figure P.15.19 shows a frame pinned to supports at A and B. The frame centreline is a circular arc and its section is uniform, of bending stiffness EI and depth d. Find the maximum stress in the frame produced by a uniform temperature gradient through the depth, the temperature on the outer and inner surfaces being raised and lowered by an amount T. The coefficient of linear expansion of the material of the frame is α. (Hint. Treat half the frame as a curved cantilever built-in on its axis of symmetry and determine the horizontal reaction at a support by equating the horizontal deflection produced by the temperature gradient to the horizontal deflection produced by the reaction).

Ans. $1.29ET\alpha$.

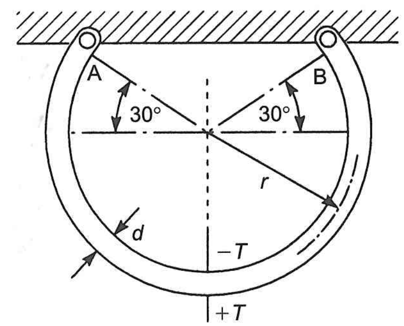


FIGURE P.15.19

P.15.20 A simply supported beam AB of span L and uniform section carries a distributed load of intensity varying from zero at A to w_o/unit length at B according to the law

$$w = \frac{2w_o x}{L}\left(1 - \frac{x}{2L}\right)$$

per unit length. If the deflected shape of the beam is given approximately by the expression

$$v = a_1 \sin\frac{\pi x}{L} + a_2 \sin\frac{2\pi x}{L}$$

evaluate the coefficients a_1 and a_2 and find the deflection of the beam at mid-span.

Ans. $a_1 = 2w_o L^4(\pi^2+4)/EI\pi^7$, $a_2 = -w_o L^4/16EI\pi^5$, $0.00918w_o L^4/EI$.

P.15.21 A uniform simply supported beam, span L, carries a distributed loading which varies according to a parabolic law across the span. The load intensity is zero at both ends of the beam and w_o at its mid-point. The loading is normal to a principal axis of the beam cross section and the relevant flexural rigidity is EI. Assuming that the deflected shape of the beam can be represented by the series

$$v = \sum_{i=1}^{\infty} a_i \sin\frac{i\pi x}{L}$$

find the coefficients a_i and the deflection at the mid-span of the beam using only the first term in this series.

Ans. $a_i = 32w_o L^4/EI\pi^7 i^7$ (i odd), $w_o L^4/94.4EI$.

P.15.22 Calculate the deflection at the mid-span point of the beam of Ex. 15.18 by assuming a deflected shape function of the form

$$v = v_1 \sin\frac{\pi x}{L} + v_3 \sin\frac{3\pi x}{L}$$

in which v_1 and v_3 are unknown displacement parameters. Note:

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{L}{2} \quad \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = 0$$

Ans. $0.02078\,WL^3/EI$.

P.15.23 A beam is supported at both ends and has the central half of its span reinforced such that its flexural rigidity is $2EI$; the flexural rigidity of the remaining parts of the beam is EI. The beam has a span L and carries a vertically downward concentrated load, W, at its mid-span point. Assuming a deflected shape function of the form

$$v = \frac{4v_m x^2}{L^3}(3L - 4x) \quad (0 \le x \le L/2)$$

in which v_m is the deflection at the mid-span point, determine the value of v_m.

Ans. $0.00347\,WL^3/EI$.

P.15.24 Figure P.15.24 shows two cantilevers, the end of one being vertically above the end of the other and connected to it by a spring AB. Initially the system is unstrained. A weight, W, placed at A causes a vertical deflection at A of δ_1 and a vertical deflection at B of δ_2. When the spring is removed the weight W at A causes a deflection at A of δ_3. Find the extension of the spring when it is replaced and the weight, W, is transferred to B.

Ans. $\delta_2(\delta_1-\delta_2)/(\delta_3-\delta_1)$

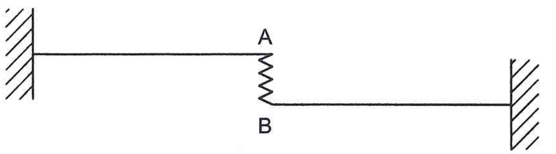


FIGURE P.15.24

P.15.25 A beam 2.4 m long is simply supported at two points A and B which are 1.44 m apart; point A is 0.36 m from the left-hand end of the beam and point B is 0.6 m from the right-hand end; the value of EI for the beam is 240×10^8 Nmm^2. Find the slope at the supports due to a load of 2 kN applied at the mid-point of AB.

Use the reciprocal theorem in conjunction with the above result to find the deflection at the mid-point of AB due to loads of 3 kN applied at each end of the beam.

Ans. 0.011, 15.8 mm.

CHAPTER 16

Analysis of Statically Indeterminate Structures

Statically indeterminate structures occur more frequently in practice than those that are statically determinate and are generally more economical in that they are stiffer and stronger. For example, a fixed beam carrying a concentrated load at mid-span has a central displacement that is one-quarter of that of a simply supported beam of the same span and carrying the same load, while the maximum bending moment is reduced by half. It follows that a smaller beam section would be required in the fixed beam case, resulting in savings in material. There are, however, disadvantages in the use of this type of beam for, as we saw in Section 13.6, the settling of a support in a fixed beam causes bending moments that are additional to those produced by the loads, a serious problem in areas prone to subsidence. Another disadvantage of statically indeterminate structures is that their analysis requires the calculation of displacements so that their cross-sectional dimensions are required at the outset. The design of such structures therefore becomes a matter of trial and error, whereas the forces in the members of a statically determinate structure are independent of member size. On the other hand, failure of, say, a member in a statically indeterminate frame would not necessarily be catastrophic since alternative load paths would be available, at least temporarily. However, the failure of a member in, say, a statically determinate truss would lead, almost certainly, to a rapid collapse.

The choice between statically determinate and statically indeterminate structures depends to a large extent upon the purpose for which a particular structure is required. As we have seen, fixed or continuous beams are adversely affected by support settlement so that the insertion of hinges at, say, points of contraflexure would reduce the structure to a statically determinate state and eliminate the problem. This procedure would not be practical in the construction of skeletal structures for high-rise buildings so that these structures are statically indeterminate. Clearly, both types of structures exist in practice so that methods of analysis are required for both statically indeterminate and statically determinate structures.

In this chapter we shall examine methods of analysis of different forms of statically indeterminate structures; as a preliminary we shall discuss the basis of the different methods, and investigate methods of determining the degree of statical and kinematic indeterminacy, an essential part of the analytical procedure.

16.1 Flexibility and stiffness methods

In Section 4.4 we briefly discussed the statical indeterminacy of trusses and established a condition, not always applicable, for a truss to be stable and statically determinate. This condition, which related the number of members and the number of joints, did not involve the support reactions which themselves could be either statically determinate or indeterminate. The condition was therefore one of *internal*

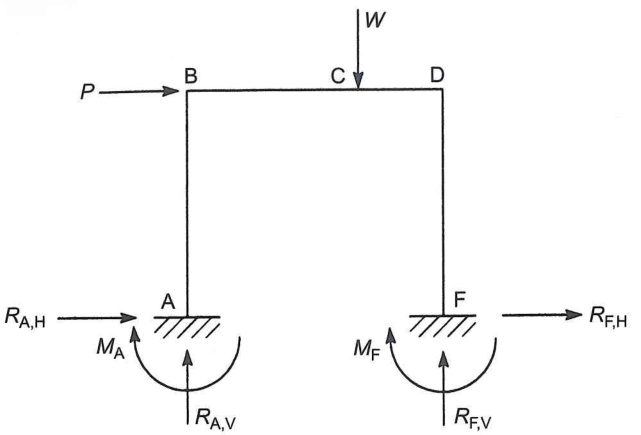


FIGURE 16.1

Statical indeterminacy of a portal frame.

statical determinacy; clearly the determinacy, or otherwise, of the support reactions is one of *external statical determinacy.*

Consider the portal frame shown in Fig. 16.1. The frame carries loads, P and W, in its own plane so that the system is two-dimensional. Since the vertical members AB and FD of the frame are fixed at A and F, the applied loads will generate a total of six reactions of force and moment as shown. For a two-dimensional system there are three possible equations of statical equilibrium (Eq. (2.10)) so that the frame is externally statically indeterminate to the *third degree.* The situation is not improved by taking a section through one of the members since this procedure, although eliminating one of the sets of reactive forces, would introduce three internal stress resultants. If, however, three of the support reactions were known or, alternatively, if the three internal stress resultants were known, the remaining three unknowns could be determined from the equations of statical equilibrium and the solution completed.

A different situation arises in the simple truss shown in Fig. 4.7(b) where, as we saw, the additional diagonal results in the truss becoming internally statically indeterminate to the *first degree;* note that the support reactions are statically determinate.

In the analysis of statically indeterminate structures two basic methods are employed. In one the structure is reduced to a statically determinate state by employing *releases,* i.e. by eliminating a sufficient number of unknowns to enable the support reactions and/or the internal stress resultants to be found from a consideration of statical equilibrium. For example, in the frame in Fig. 16.1 the number of support reactions would be reduced to three if one of the supports was pinned and the other was a pinned roller support. The same result would be achieved if one support remained fixed and the other support was removed entirely. Also, in the truss in Fig. 4.7(b), removing a diagonal, vertical or horizontal member would result in the truss becoming statically determinate. Releasing a structure in this way would produce displacements that would not otherwise be present. These displacements may be calculated by analysing the released statically determinate structure; the force system required to eliminate them is then obtained, i.e. we are employing a compatibility of displacement condition. This method is generally termed the *flexibility* or *force method;* in effect this method was used in the solution of the propped cantilever in Fig. 13.25.

The alternative procedure, known as the *stiffness* or *displacement method* is analogous to the flexibility method, the major difference being that the unknowns are the displacements at specific points in the structure. Generally the procedure requires a structure to be divided into a number of elements for each of which load–displacement relationships are known. Equations of equilibrium are then written down in terms of the displacements at the element junctions and are solved for the required displace-

Both the flexibility and stiffness methods generally result, for practical structures having a high degree of statical indeterminacy, in a large number of simultaneous equations which are most readily solved by computer-based techniques. However, the flexibility method requires the structure to be reduced to a statically determinate state by inserting releases, a procedure requiring some judgement on the part of the analyst. The stiffness method, on the other hand, requires no such judgement to be made and is therefore particularly suitable for automatic computation.

Although the practical application of the flexibility and stiffness methods is generally computer based, they are fundamental to 'hand' methods of analysis as we shall see. Before investigating these hand methods we shall examine in greater detail the indeterminacy of structures since we shall require the degree of indeterminacy of a structure before, in the case of the flexibility method, the appropriate number of releases can be determined. At the same time the *kinematic indeterminacy* of a structure is needed to determine the number of constraints that must be applied to render the structure kinematically determinate in the stiffness method.

16.2 Degree of statical indeterminacy

In some cases the degree of statical indeterminacy of a structure is obvious from inspection. For example, the portal frame in Fig. 16.1 has a degree of external statical indeterminacy of 3, while the truss of Fig. 4.7(b) has a degree of internal statical indeterminacy of 1. However, in many cases, the degree is not obvious and in other cases the internal and external indeterminacies may not be independent so that we need to consider the complete structure, including the support system. A more formal and methodical approach is therefore required.

Rings

The simplest approach is to insert constraints in a structure until it becomes a series of completely stiff *rings.* The statical indeterminacy of a ring is known and hence that of the completely stiff structure. Then by inserting the number of releases required to return the completely stiff structure to its original state, the degree of indeterminacy of the actual structure is found.

Consider the single ring shown in Fig. 16.2(a); the ring is in equilibrium in space under the action of a number of forces that are not coplanar. If, say, the ring is cut at some point, X, the cut ends of the ring will be displaced relative to each other as shown in Fig. 16.2(b) since, in effect, the internal forces equilibrating the external forces have been removed. The cut ends of the ring will move relative to each other in up to six possible ways until a new equilibrium position is found, i.e. translationally along the x, y and z axes and rotationally about the x, y and z axes, as shown in Fig. 16.2(c). The ring is now statically determinate and the internal force system at any section may be obtained from simple equilibrium considerations. To rejoin the ends of the ring we require forces and moments proportional to the displacements, i.e. three forces and three moments. Therefore at any section in a complete ring subjected to an arbitrary external loading system there are three internal forces and three internal moments, none of which may be obtained by statics. A ring is then six times statically indeterminate. For a two-dimensional system in which the forces are applied in the plane of the ring, the internal force system is reduced to an axial force, a shear force and a moment, so that a two-dimensional ring is three times statically indeterminate.

The above arguments apply to any closed loop so that a ring may be of any shape. Furthermore, a ring may be regarded as comprising any number of members which form a closed loop and which are joined at *nodes,* a node being defined as a point at the end of a member. Examples of rings are shown

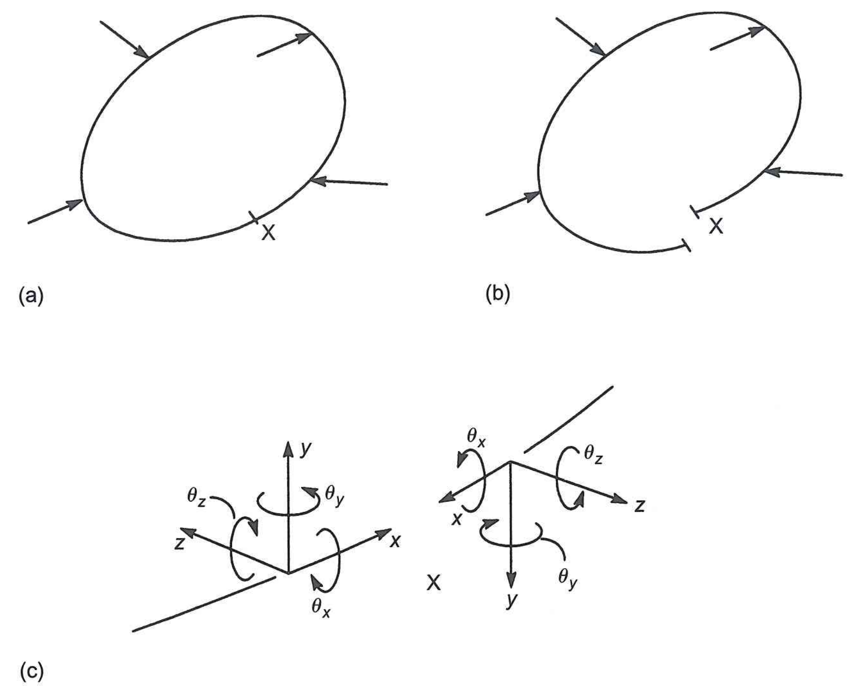


FIGURE 16.2

Statical indeterminacy of a ring.

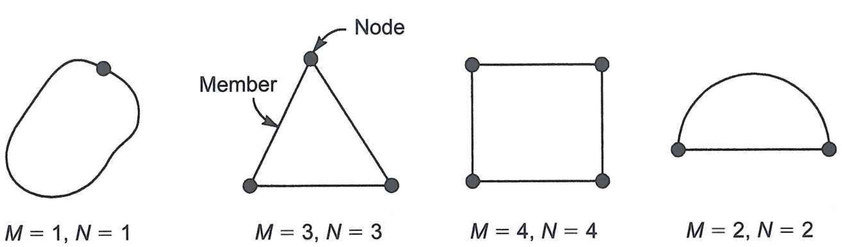


FIGURE 16.3

Examples of rings.

number of members is equal to the number of nodes in every case. However, when a ring is cut we introduce an additional member and two additional nodes, as shown in Fig. 16.4.

The entire structure

Since we shall require the number of rings in a structure, and since it is generally necessary to include the support system, we must decide what constitutes the structure. For example, in Fig. 16.5 the members AB and BC are pinned to the foundation at A and C. The foundation therefore acts as a member of very high stiffness. In this simple illustration it is obvious that the members AB and BC, with the foundation, form a ring if the pinned joints are replaced by rigid joints. In more complex structures we must ensure that just sufficient of the foundation is included so that superfluous indeterminacies are

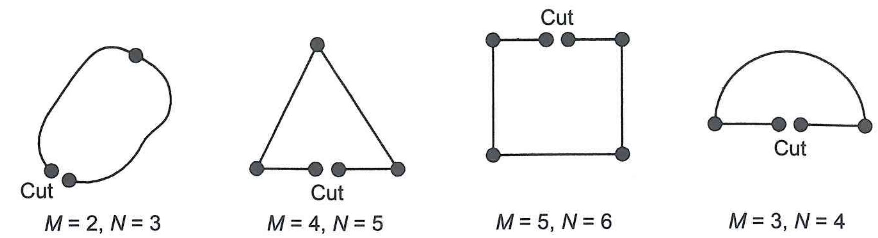


FIGURE 16.4

Effect on members and nodes of cutting a ring.

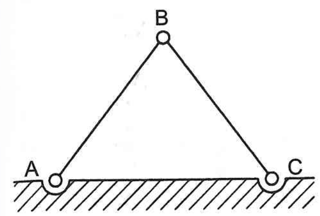


FIGURE 16.5

Foundation acting as a structural member.

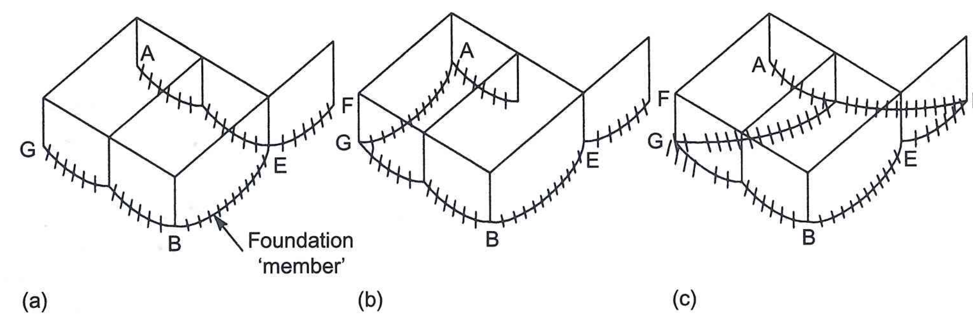


FIGURE 16.6

Determination of the entire structure.

of support are *singly connected* such that for any two points A and B in the foundation system there is only one path from A to B that does not involve retracing any part of the path. For example, in Fig. 16.6(a) and (b) there is only one path between A and B which does not involve retracing part of the path. In Fig. 16.6(c), however, there are two possible paths from A to B, one via G and one via F and E. Thus the support points in Fig. 16.6(a) and (b) are singly connected while those in Fig. 16.6(c) are multiply connected. We note from the above that there may be a number of ways of singly connecting the support points in a foundation system and that each support point in the entire structure is attached to at least one foundation 'member'. Including the foundation members increases the number of members, but the number of nodes is unchanged.

The completely stiff structure

Having established the entire structure we now require the *completely stiff structure* in which there is no

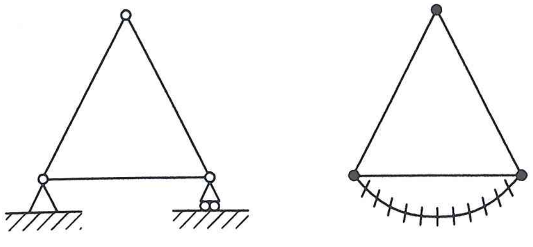


FIGURE 16.7

A completely stiff structure.

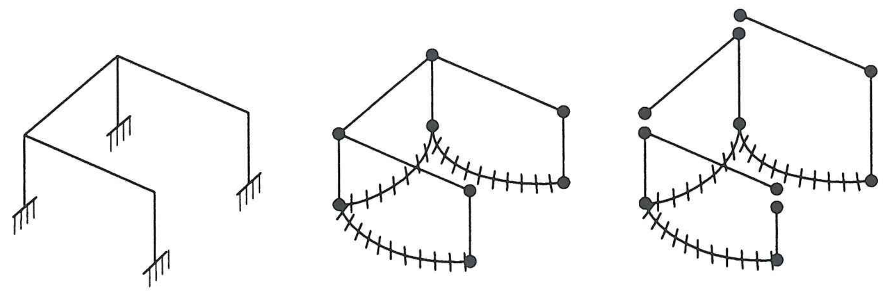


FIGURE 16.8

Determination of the degree of statical indeterminacy of a structure.

completely stiff structure (Fig. 16.7(b)) corresponding to the simple truss in Fig. 16.7(a) has rigid joints (nodes), members that are capable of resisting shear loads as well as axial loads and a single foundation member. Note that the completely stiff structure comprises two rings, is two-dimensional and therefore six times statically indeterminate. We shall consider how such a structure is 'released' to return it to its original state (Fig. 16.7(a)) after considering the degree of indeterminacy of a three-dimensional system.

Degree of statical indeterminacy

Consider the frame structure shown in Fig. 16.8(a). It is three-dimensional and comprises three portal frames that are rigidly built-in at the foundation. Its completely stiff equivalent is shown in Fig. 16.8 (b) where we observe by inspection that it consists of three rings, each of which is six times statically indeterminate so that the completely stiff structure is $3 \times 6 = 18$ times statically indeterminate. Although the number of rings in simple cases such as this is easily found by inspection, more complex cases require a more methodical approach.

Suppose that the members are disconnected until the structure becomes singly connected as shown in Fig. 16.8(c). (A singly connected structure is defined in the same way as a singly connected foundation.) Each time a member is disconnected, the number of nodes increases by one, while the number of rings is reduced by one; the number of members remains the same. The final number of nodes, N′, in the singly connected structure is therefore given by

$$N' = M + 1 \ (M = \text{number of members})$$

Suppose now that the members are reconnected to form the original completely stiff structure. Each reconnection forms a ring, i.e. each time a node disappears a ring is formed so that the number of rings, R, is equal to the number of nodes lost during the reconnection. Thus

$$R = N' - N$$

where N is the number of nodes in the completely stiff structure. Substituting for N' from the above we have

$$R = M - N + 1$$

In Fig. 16.8(b), $M = 10$ and $N = 8$ so that $R = 3$ as deduced by inspection. Therefore, since each ring is six times statically indeterminate, the degree of statical indeterminacy, n_s', of the completely stiff structure is given by

$$n_s' = 6(M - N + 1) \tag{16.1}$$

For an actual entire structure, releases must be inserted to return the completely stiff structure to its original state. Each release will reduce the statical indeterminacy by 1, so that if r is the total number of releases required, the degree of statical indeterminacy, n_s, of the actual structure is

$$n_s = n_s' - r$$

or, substituting for n_s' from Eq. (16.1)

$$n_s = 6(M - N + 1) - r \tag{16.2}$$

Note that in Fig. 16.8 no releases are required to return the completely stiff structure of Fig. 16.8 (b) to its original state in Fig. 16.8(a) so that its degree of indeterminacy is 18.

In the case of two-dimensional structures in which a ring is three times statically indeterminate, Eq. (16.2) becomes

$$n_s = 3(M - N + 1) - r \tag{16.3}$$

Trusses

A difficulty arises in determining the number of releases required to return the completely stiff equivalent of a truss to its original state.

Consider the completely stiff equivalent of a plane truss shown in Fig. 16.9(a); we are not concerned here with the indeterminacy or otherwise of the support system which is therefore omitted. In the actual truss each member is assumed to be capable of resisting axial load only so that there are two releases for each member, one of shear and one of moment, a total of $2M$ releases. Thus, if we insert a hinge at the end of each member as shown in Fig. 16.9(b) we have achieved the required number, $2M$, of releases. However,

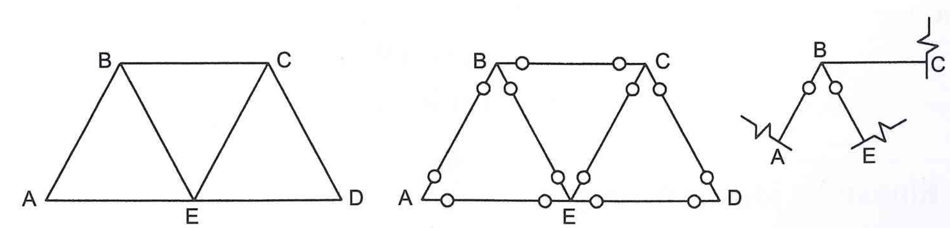


FIGURE 16.9

in this configuration, each joint would be free to rotate as a mechanism through an infinitesimally small angle, independently of the members; the truss is then excessively pin-jointed. This situation can be prevented by removing one hinge at each joint as shown, for example at joint B in Fig. 16.9(c). The member BC then prevents rotation of the joint at B. Furthermore, the presence of a hinge at B in BA and at B in BE ensures that there is no moment at B in BC so that the conditions for a truss are satisfied.

From the above we see that the total number, $2M$, of releases is reduced by 1 for each node. Thus the required number of releases in a plane truss is

$$r = 2M - N \tag{16.4}$$

so that Eq. (16.3) becomes

$$n_s = 3(M - N + 1) - (2M - N)$$

or

$$n_s = M - 2N + 3 \tag{16.5}$$

Equation (16.5) refers only to the internal indeterminacy of a truss so that the degree of indeterminacy of the support system is additional. Also, returning to the simple triangular truss of Fig. 16.7(a) we see that its degree of internal indeterminacy is, from Eq. (16.5), given by

$$n_s = 3 - 2 \times 3 + 3 = 0$$

as expected.

A similar situation arises in a space truss where, again, each member is required to resist axial load only so that there are $5M$ releases for the complete truss. This could be achieved by inserting ball joints at the ends of each member. However, we would then be in the same kind of position as the plane truss of Fig. 16.9(b) in that each joint would be free to rotate through infinitesimally small angles about each of the three axes (the members in the plane truss can only rotate about one axis) so that three constraints are required at each node, a total of $3N$ constraints. Therefore the number of releases is given by

$$r = 5M - 3N$$

so that Eq. (16.2) becomes

$$n_s = 6(M - N + 1) - (5M - 3N)$$

or

$$n_s = M - 3N + 6 \tag{16.6}$$

For statically determinate plane trusses and space trusses, i.e. $n_s = 0$, Eqs (16.5) and (16.6), respectively, become

$$M = 2N - 3 \quad M = 3N - 6 \tag{16.7}$$

which are the results deduced in Section 4.4 (Eqs (4.1) and (4.2)).

16.3 Kinematic indeterminacy

We have seen that the degree of statical indeterminacy of a structure is, in fact, the number of forces or stress resultants which cannot be determined using the equations of statical equilibrium. Another form of the indeterminacy of a structure is expressed in terms of its *degrees of freedom;* this is known as the

A simple approach to calculating the kinematic indeterminacy of a structure is to sum the degrees of freedom of the nodes and then subtract those degrees of freedom that are prevented by constraints such as support points. It is therefore important to remember that in three-dimensional structures each node possesses 6 degrees of freedom while in plane structures each node possesses three degrees of freedom.

EXAMPLE 16.1

Determine the degrees of statical and kinematic indeterminacy of the beam ABC shown in Fig. 16.10(a).

The completely stiff structure is shown in Fig. 16.10(b) where we see that $M = 4$ and $N = 3$. The number of releases, r, required to return the completely stiff structure to its original state is 5, as indicated in Fig. 16.10(b); these comprise a moment release at each of the three supports and a translational release at each of the supports B and C. Therefore, from Eq. (16.3)

$$n_s = 3(4 - 3 + 1) - 5 = 1$$

so that the degree of statical indeterminacy of the beam is 1.

Each of the three nodes possesses 3 degrees of freedom, a total of nine. There are four constraints so that the degree of kinematic indeterminacy is given by

$$n_k = 9 - 4 = 5$$

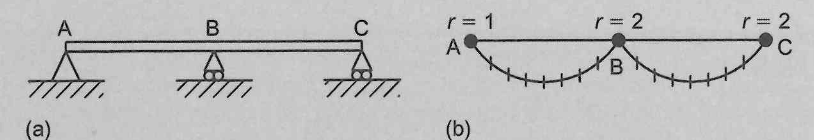


FIGURE 16.10

Determination of the statical and kinematic indeterminacies of the beam of Ex. 16.1.

EXAMPLE 16.2

Determine the degree of statical and kinematic indeterminacy of the truss shown in Fig. 16.11(a).

The completely stiff structure is shown in Fig. 16.11(b) in which we see that $M = 17$ and $N = 8$. However, since the truss is pin-jointed, we can obtain the internal statical indeterminacy directly from Eq. (16.5) in which $M = 16$, the actual number of truss members. Thus

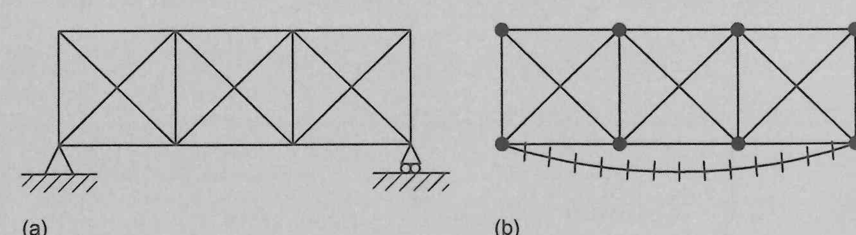


FIGURE 16.11

Determinacy of the truss of Ex. 16.2.

$$n_s = 16 - 16 + 3 = 3$$

and since, as can be seen from inspection, the support system is statically determinate, the complete structure is three times statically indeterminate.

Alternatively, considering the completely stiff structure in Fig. 16.11(b) in which $M = 17$ and $N = 8$, we can use Eq. (16.3). The number of internal releases is found from Eq. (16.4) and is $r = 2 \times 16 - 8 = 24$. There are three additional releases from the support system giving a total of 27 releases. Thus, from Eq. (16.3)

$$n_s = 3(17 - 8 + 1) - 27 = 3$$

as before.

The kinematic indeterminacy is found as before by examining the total degrees of freedom of the nodes and the constraints, which in this case are provided solely by the support system. There are eight nodes each having 2 translational degrees of freedom. The rotation at a node does not result in a stress resultant and is therefore irrelevant. There are therefore 2 degrees of freedom at a node in a plane truss and 3 in a space truss. In this example there are then $8 \times 2 = 16$ degrees of freedom and three translational constraints from the support system. Thus

$$n_k = 16 - 3 = 13$$

EXAMPLE 16.3

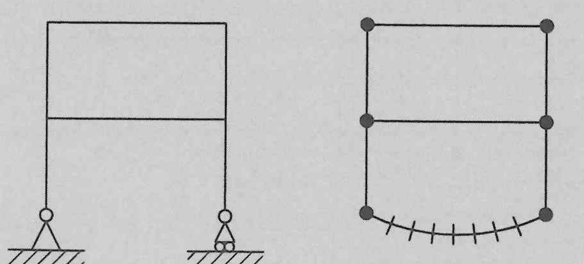
Calculate the degree of statical and kinematic indeterminacy of the frame shown in Fig. 16.12(a).

In the completely stiff structure shown in Fig. 16.12(b), $M = 7$ and $N = 6$. The number of releases, r, required to return the completely stiff structure to its original state is 3. Thus, from Eq. (16.3)

$$n_s = 3(7 - 6 + 1) - 3 = 3$$

The number of nodes is six, each having 3 degrees of freedom, a total of 18. The number of constraints is three so that the kinematic indeterminacy of the frame is given by

$$n_k = 18 - 3 = 15$$



(a) (b)

FIGURE 16.12

Statical and kinematic indeterminacies of the frame of Ex. 16.3.

EXAMPLE 16.4

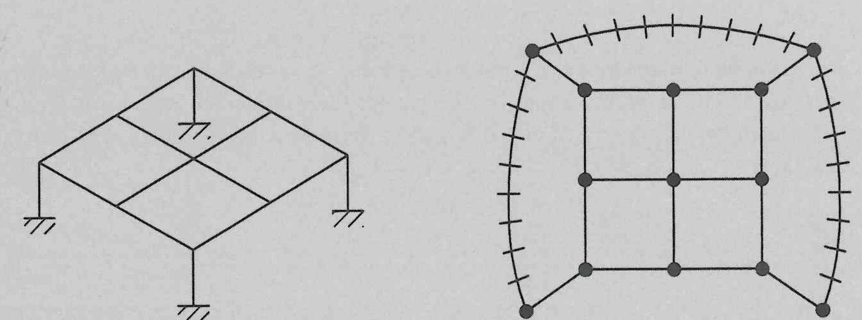
Determine the degree of statical and kinematic indeterminacy in the space frame shown in Fig. 16.13(a).

In the completely stiff structure shown in Fig. 16.13(b), $M = 19$, $N = 13$ and $r = 0$. Therefore from Eq. (16.2)

$$n_s = 6(19 - 13 + 1) - 0 = 42$$

There are 13 nodes each having 6 degrees of freedom, a total of 78. There are six constraints at each of the four supports, a total of 24. Thus

$$n_k = 78 - 24 = 54$$



(a) (b)

FIGURE 16.13

Space frame of Ex. 16.4.

We shall now consider different types of statically indeterminate structure and the methods that may be used to analyse them; the methods are based on the work and energy methods described in Chapter 15.

16.4 Statically indeterminate beams

Beams are statically indeterminate generally because of their support systems. In this category are propped cantilevers, fixed beams and continuous beams. A propped cantilever and some fixed beams were analysed in Section 13.6 using either the principle of superposition or moment-area methods. We shall now apply the methods described in Chapter 15 to some examples of statically indeterminate beams.

EXAMPLE 16.5

Calculate the support reaction at B in the propped cantilever shown in Fig. 16.14.

In this example it is unnecessary to employ the procedures described in Section 16.2 to calculate the degree of statical indeterminacy since this is obvious by inspection. Thus the removal of the vertical support at B would result in a statically determinate cantilever beam so that we deduce that the degree of statical indeterminacy is 1. Furthermore, it is immaterial whether we use the principle of

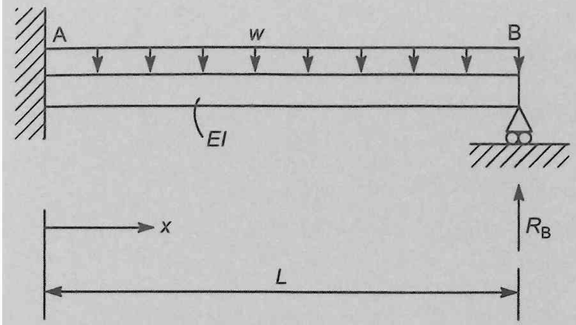


FIGURE 16.14

Propped cantilever of Ex. 16.5.

virtual work or complementary energy in the solution since, for linearly elastic systems, they result in the same equations (see Chapter 15). First, we shall adopt the complementary energy approach.

The total complementary energy, C, of the beam is given, from Eq. (i) of Ex. 15.13, by

$$C = \int_0^L \int_0^M d\theta \, dM - R_B v_B \qquad \text{(i)}$$

in which v_B is the vertical displacement of the cantilever at B (in this case $v_B = 0$ since the beam is supported at B).

From the principle of the stationary value of the total complementary energy we have

$$\frac{\partial C}{\partial R_B} = \int_0^L \frac{\partial M}{\partial R_B} d\theta - v_B = 0 \qquad \text{(ii)}$$

which, by comparison with Eq. (iii) of Ex. 15.13, becomes

$$v_B = \int_0^L \frac{M}{EI} \frac{\partial M}{\partial R_B} dx = 0 \qquad \text{(iii)}$$

The bending moment, M, at any section of the beam is given by

$$M = R_B(L - x) - \frac{w}{2}(L-x)^2$$

Hence

$$\frac{\partial M}{\partial R_B} = L - x$$

Substituting in Eq. (iii) for M and $\partial M/\partial R_B$ we have

$$\int_0^L \left\{ R_B(L - x)^2 - \frac{w}{2}(L - x)^3 \right\} dx = 0 \qquad \text{(iv)}$$

from which

$$R_B = \frac{3wL}{8}$$

which is the result obtained in Ex. 13.20.

The algebra in the above solution would have been slightly simplified if we had assumed an origin for x at the end B of the beam. Equation (iv) would then become

$$\int_0^L \left(R_B x^2 - \frac{w}{2} x^3 \right) dx = 0$$

which again gives

$$R_B = \frac{3wL}{8}$$

Having obtained R_B, the remaining support reactions follow from statics.

An alternative approach is to release the structure so that it becomes statically determinate by removing the support at B (by inspection the degree of statical indeterminacy is 1 so that one release only is required in this case) and then to calculate the vertical displacement at B due to the applied load using, say, the unit load method which, as we have seen, is based on the principle of virtual work or, alternatively, complementary energy. We then calculate the vertical displacement at B produced by R_B acting alone, again, say, by the unit load method. The sum of the two displacements must be zero since the beam at B is supported, so that we obtain an equation in which R_B is the unknown.

It is not essential to select the support reaction at B as the release. We could, in fact, choose the fixing moment at A in which case the beam would become a simply supported beam which, of course, is statically determinate. We would then determine the moment at A required to restore the slope of the beam at A to zero.

In the above, the released structure is frequently termed the *primary structure.*

Suppose that the vertical displacement at the free end of the released cantilever due to the uniformly distributed load is $v_{B,0}$. Then, from Eq. (iii) of Ex. 15.14 (noting that M_A in that equation has been replaced by M_a here to avoid confusion with the bending moment at A)

$$v_{B,0} = \int_0^L \frac{M_a M_1}{EI} dx \qquad \text{(v)}$$

in which

$$M_a = -\frac{w}{2}(L - x)^2 \quad M_1 = -1(L - x)$$

Hence, substituting for M_a and M_1 in Eq. (v), we have

$$v_{B,0} = \int_0^L \frac{w}{2EI}(L - x)^3 dx$$

which gives

$$v_{B,0} = \frac{wL^4}{8EI} \qquad \text{(vi)}$$

We now apply a vertically downward unit load at the B end of the cantilever from which the distributed load has been removed. The displacement, $v_{B,1}$, due to this unit load is then, from Eq. (v)

$$v_{B,1} = \int_0^L \frac{1}{EI}(L - x)^2 dx$$

from which

$$v_{B,1} = \frac{L^3}{3EI} \tag{vii}$$

The displacement due to R_B at B is $-R_B v_{B,1}$ (R_B acts in the opposite direction to the unit load) so that the total displacement, v_B, at B due to the uniformly distributed load and R_B is, using the principle of superposition

$$v_B = v_{B,0} - R_B v_{B,1} = 0 \tag{viii}$$

Substituting for $v_{B,0}$ and $v_{B,1}$ from Eqs (vi) and (vii) we have

$$\frac{wL^4}{8EI} - R_B \frac{L^3}{3EI} = 0$$

which gives

$$R_B = \frac{3wL}{8}$$

as before. This approach is the flexibility method described in Section 16.1 and is, in effect, identical to the method used in Ex. 13.20.

In Eq. (viii) $v_{B,1}$ is the displacement at B in the direction of R_B due to a unit load at B applied in the direction of R_B (either in the same or opposite directions). For a beam that has a degree of statical indeterminacy greater than 1 there will be a series of equations of the same form as Eq. (viii) but which will contain the displacements at a specific point produced by the redundant forces. We shall therefore employ the *flexibility coefficient* a_{kj} ($k = 1,2,\ldots, r$; $j = 1, 2,\ldots, r$) which we defined in Section 15.4 as the displacement at a point k in a given direction produced by a unit load at a point j in a second direction. Thus, in the above, $v_{B,1} = a_{11}$ so that Eq. (viii) becomes

$$v_{B,0} - a_{11} R_B = 0 \tag{ix}$$

It is also convenient, since the flexibility coefficients are specified by numerical subscripts, to redesignate R_B as R_1. Thus Eq. (ix) becomes

$$v_{B,0} - a_{11} R_1 = 0 \tag{x}$$

EXAMPLE 16.6

Determine the support reaction at B in the propped cantilever shown in Fig. 16.15(a).

As in Ex. 16.5, the cantilever in Fig. 16.15(a) has a degree of statical indeterminacy equal to 1. Again we shall choose the support reaction at B, R_1, as the indeterminacy; the released or primary structure is shown in Fig. 16.15(b). Initially we require the displacement, $v_{B,0}$, at B due to the applied load, W, at C. This may readily be found using the unit load method. Thus from Eq. (iii) of Ex. 15.13

$$v_{B,0} = \int_0^L \left\{-\frac{W}{EI}\left(\frac{3L}{2} - x\right)\right\}\{-1(L-x)\}\mathrm{d}x$$

which gives

$$7WL^3$$

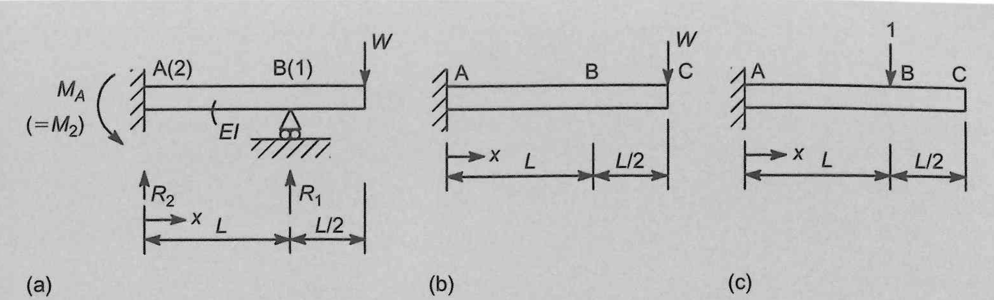


FIGURE 16.15

Propped cantilever of Ex. 16.6 (a).

Similarly, the displacement at B due to the unit load at B in the direction of R_1 (Fig. 16.15(c)) is

$$a_{11} = \frac{L^3}{3EI} \quad \text{(use Eq. (vii) of Ex. 16.5)}$$

Hence, since,

$$v_{B,0} - a_{11} R_1 = 0 \tag{ii}$$

we have

$$\frac{7WL^3}{12EI} - \frac{L^3}{3EI} R_1 = 0$$

from which

$$R_1 = \frac{7W}{4}$$

Alternatively, we could select the fixing moment, M_A ($=M_2$), at A as the release. The primary structure is then the simply supported beam shown in Fig. 16.16(a) where $R_A = -W/2$ and $R_B = 3W/2$. The rotation at A may be found by any of the methods previously described. They include the integration of the second-order differential equation of bending (Eq. (13.3)), the moment-area method described in Section 13.3 and the unit load method (in this case it would be a unit moment). Thus, using the unit load method and applying a unit moment at A as shown in Fig. 16.16(b) we have, from the principle of virtual work (see Eq. (i) of "Ex. 15.8")

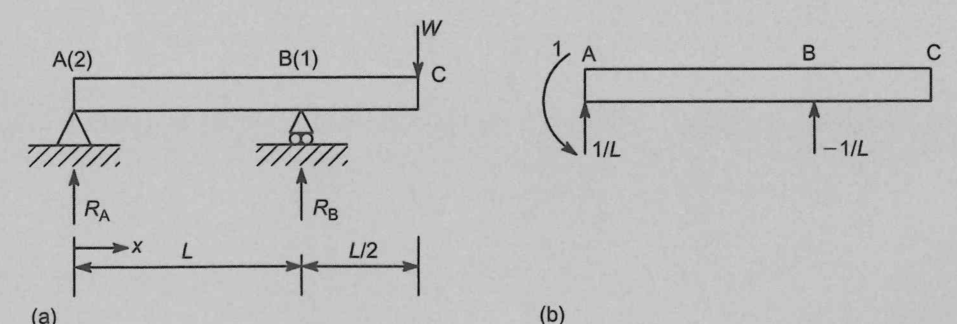


FIGURE 16.16

$$1\theta_{A,0} = \int_0^L \frac{M_a M_v}{EI}\,dx + \int_L^{3L/2} \frac{M_a M_v}{EI}\,dx \qquad \text{(iii)}$$

In Eq. (iii)

$$M_a = -\frac{W}{2}x \quad M_v = \frac{1}{L}x - 1 \quad (0 \le x \le L)$$

$$M_a = Wx - \frac{3WL}{2} \quad M_v = 0 \quad \left(L \le x \le \frac{3L}{2}\right)$$

Substituting in Eq. (iii) we have

$$\theta_{A,0} = \frac{W}{2EIL}\int_0^L (Lx - x^2)\,dx$$

from which

$$\theta_{A,0} = \frac{WL^2}{12EI} \quad \text{(anticlockwise)}$$

The flexibility coefficient, θ_{22}, i.e. the rotation at A (point 2), due to a unit moment at A is obtained from Fig. 16.16(b). Thus

$$\theta_{22} = \int_0^L \frac{1}{EI}\left(\frac{x}{L} - 1\right)^2 dx$$

from which

$$\theta_{22} = \frac{L}{3EI} \quad \text{(anticlockwise)}$$

Therefore, since the rotation at A in the actual structure is zero

$$\theta_{A,0} + \theta_{22}M_2 = 0$$

or

$$\frac{WL^2}{12EI} + \frac{L}{3EI}M_2 = 0$$

which gives

$$M_2 = -\frac{WL}{4} \quad \text{(clockwise)}$$

Considering now the statical equilibrium of the beam in Fig. 16.15(a) we have, taking moments about A

$$R_1 L - W\frac{3L}{2} - \frac{WL}{4} = 0$$

so that

$$R_1 = \frac{7WL}{4}$$

as before.

EXAMPLE 16.7

Determine the support reactions in the three-span continuous beam ABCD shown in Fig. 16.17(a).

It is clear from inspection that the degree of statical indeterminacy of the beam is two. Therefore, if we choose the supports at B and C as the releases, the primary structure is that shown in Fig. 16.17(b). We therefore require the vertical displacements, $v_{B,0}$ and $v_{C,0}$, at the points B and C. These may readily be found using any of the methods previously described (unit load method, moment-area method, Macauley's method (Section 13.2)) and are

$$v_{B,0} = \frac{8.88}{EI} \quad v_{C,0} = \frac{9.08}{EI} \quad \text{(downwards)}$$

We now require the flexibility coefficients, a_{11}, a_{12}, a_{22} and a_{21}. The coefficients a_{11} and a_{21} are found by placing a unit load at B (point 1) as shown in Fig. 16.17(c) and then determining the displacements at B and C (point 2). Similarly, the coefficients a_{22} and a_{12} are found by placing a unit load at C and calculating the displacements at C and B; again, any of the methods listed above may be used. However, from the reciprocal theorem (Section 15.4) $a_{12} = a_{21}$ and from symmetry $a_{11} = a_{22}$. Therefore it is only necessary to calculate the displacements a_{11} and a_{21} from Fig. 16.17(c). These are

$$a_{11} = a_{22} = \frac{0.45}{EI} \quad a_{21} = a_{12} = \frac{0.39}{EI} \quad \text{(downwards)}$$

The total displacements at the support points B and C are zero so that

$$v_{B,0} - a_{11}R_1 - a_{12}R_2 = 0 \qquad \text{(i)}$$

$$v_{C,0} - a_{21}R_1 - a_{22}R_2 = 0 \qquad \text{(ii)}$$

or, substituting the calculated values of $v_{B,0}$, a_{11}, etc., in Eqs (i) and (ii), and multiplying through by EI

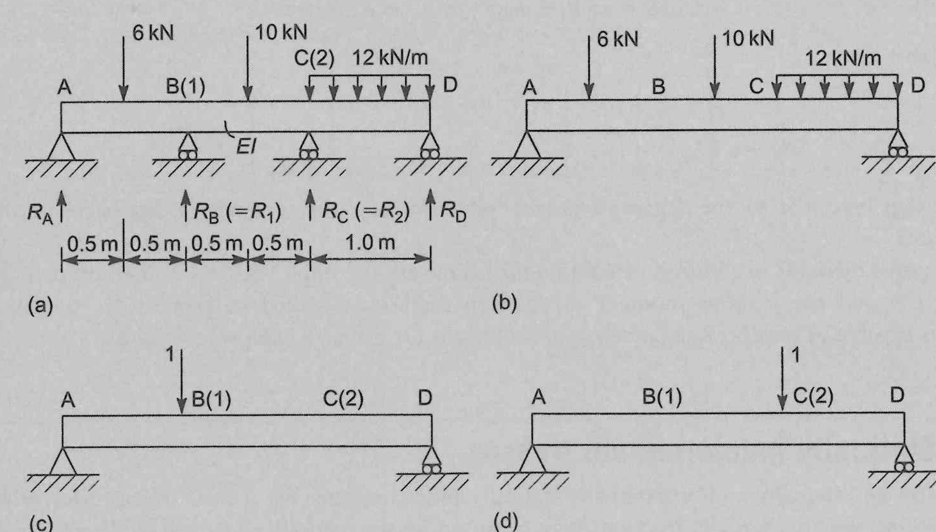


FIGURE 16.17

Analysis of a three-span continuous beam

$$8.88 - 0.45R_1 - 0.39R_2 = 0 \qquad \text{(iii)}$$

$$9.08 - 0.39R_1 - 0.45R_2 = 0 \qquad \text{(iv)}$$

Note that the negative signs in the terms involving R_1 and R_2 in Eqs (i) and (ii) are due to the fact that the unit loads were applied in the opposite directions to R_1 and R_2. Solving Eqs (iii) and (iv) we obtain

$$R_1(=R_B) = 8.7 \text{ kN} \quad R_2(=R_C) = 12.68 \text{ kN}$$

The remaining reactions are determined by considering the statical equilibrium of the beam and are

$$R_A = 1.97 \text{ kN} \quad R_B = 4.65 \text{ kN}$$

In Exs 16.5–16.7 we have assumed that the beam supports are not subjected to a vertical displacement themselves. It is possible, as we have previously noted, that a support may sink, so that the right-hand side of the compatibility equations, Eqs (viii), (ix) and (x) in Ex. 16.5, Eq. (ii) in Ex. 16.6 and Eqs (i) and (ii) in Ex. 16.7, would not be zero but equal to the actual displacement of the support. In such a situation one of the releases should coincide with the displaced support.

It is clear from Ex. 16.7 that the number of simultaneous equations of the form of Eqs (i) and (ii) requiring solution is equal to the degree of statical indeterminacy of the structure. For structures possessing a high degree of statical indeterminacy the solution, by hand, of a large number of simultaneous equations is not practicable. The equations would then be expressed in matrix form and solved using a computer-based approach. Thus for a structure having a degree of statical indeterminacy equal to n there would be n compatibility equations of the form

$$\begin{aligned} v_{1,0} + a_{11}R_1 + a_{12}R_2 + \cdots + a_{1n}R_n &= 0 \\ &\vdots \\ v_{n,0} + a_{n1}R_1 + a_{n2}R_2 + \cdots + a_{nn}R_n &= 0 \end{aligned}$$

or, in matrix form

$$\begin{Bmatrix} v_{1,0} \\ \vdots \\ v_{n,0} \end{Bmatrix} = -\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \vdots & & \\ a_{n,1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{Bmatrix} R_1 \\ \vdots \\ R_n \end{Bmatrix}$$

Note that here n is n_s, the degree of statical indeterminacy; the subscript 's' has been omitted for convenience.

Alternative methods of solution of continuous beams are the slope–deflection method described in Section 16.9 and the iterative moment distribution method described in Section 16.10. The latter method is capable of producing relatively rapid solutions for beams having several spans.

16.5 Statically indeterminate trusses

A truss may be internally and/or externally statically indeterminate. For a truss that is externally statically indeterminate, the support reactions may be found by the methods described in Section 16.4. For a truss that is internally statically indeterminate the flexibility method may be employed as illustrated in the following examples.

EXAMPLE 16.8

Determine the forces in the members of the truss shown in Fig. 16.18(a); the cross-sectional area, A, and Young's modulus, E, are the same for all members.

The truss in Fig. 16.18(a) is clearly externally statically determinate but, from Eq. (16.5), has a degree of internal statical indeterminacy equal to 1 ($M = 6$, $N = 4$). We therefore release the truss so that it becomes statically determinate by 'cutting' one of the members, say BD, as shown in Fig. 16.18(b). Due to the actual loads (P in this case) the cut ends of the member BD will separate or come together, depending on whether the force in the member (before it was cut) was tensile or compressive; we shall assume that it was tensile.

We are assuming that the truss is linearly elastic so that the relative displacement of the cut ends of the member BD (in effect the movement of B and D away from or towards each other along the diagonal BD) may be found using, say, the unit load method as illustrated in Exs 15.9 and 15.12. Thus we determine the forces $F_{a,j}$ in the members produced by the actual loads. We then apply equal and opposite unit loads to the cut ends of the member BD as shown in Fig. 16.18(c) and calculate the forces, $F_{1,j}$ in the members. The displacement of B relative to D, Δ_{BD}, is then given by

$$\Delta_{BD} = \sum_{j=1}^{n} \frac{F_{a,j}F_{1,j}L_j}{AE} \qquad \text{(see Eq. (iii) in Ex. 15.12)}$$

The forces, $F_{a,j}$, are the forces in the members of the released truss due to the actual loads and are not, therefore, the actual forces in the members of the complete truss. We shall therefore redesignate the forces in the members of the released truss as $F_{0,j}$. The expression for Δ_{BD} then becomes

$$\Delta_{BD} = \sum_{j=1}^{n} \frac{F_{0,j}F_{1,j}L_j}{AE} \qquad \text{(i)}$$

In the actual structure this displacement is prevented by the force, X_{BD}, in the redundant member BD. If, therefore, we calculate the displacement, a_{BD}, in the direction of BD produced by a unit value of X_{BD}, the displacement due to X_{BD} will be $X_{BD}a_{BD}$. Clearly, from compatibility

$$\Delta_{BD} + X_{BD}a_{BD} = 0 \qquad \text{(ii)}$$

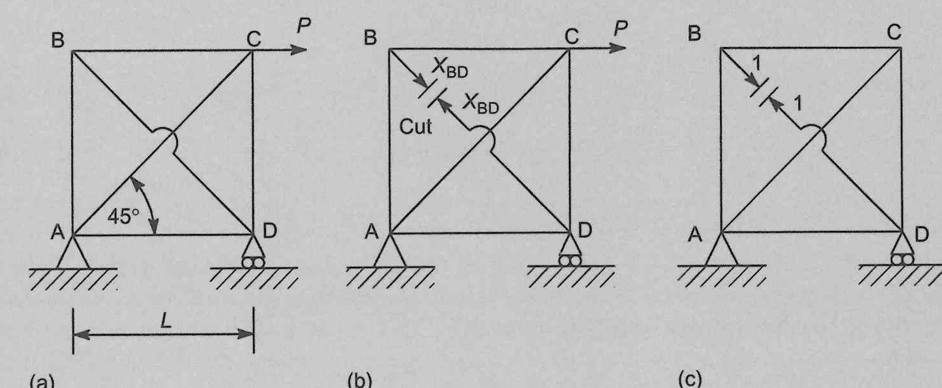


FIGURE 16.18

Analysis of a statically indeterminate truss.

Table 16.1

Member	L_j (m)	$F_{0,j}$	$F_{1,j}$	$F_{0,j}F_{1,j}L_j$	$F_{1,j}^2 L_j$	$F_{a,j}$
AB	L	0	−0.71	0	0.5L	+0.40P
BC	L	0	−0.71	0	0.5L	+0.40P
CD	L	−P	−0.71	0.71PL	0.5L	−0.60P
BD	1.41L	–	1.0	–	1.41L	−0.56P
AC	1.41L	1.41P	1.0	2.0PL	1.41L	+0.85P
AD	L	0	−0.71	0	0.5L	+0.40P
				$\Sigma = 2.71PL$	$\Sigma = 4.82L$	

from which X_{BD} is found. Again, as in the case of statically indeterminate beams, a_{BD} is a flexibility coefficient. Having determined X_{BD}, the actual forces in the members of the complete truss may be calculated by, say, the method of joints or the method of sections.

In Eq. (ii), a_{BD} is the displacement of the released truss in the direction of BD produced by a unit load. Thus, in using the unit load method to calculate this displacement, the actual member forces ($F_{1,j}$) and the member forces produced by the unit load ($F_{1,j}$) are the same. Therefore, from Eq. (i)

$$a_{BD} = \sum_{j=1}^{n} \frac{F_{1,j}^2 L_j}{AE} \qquad \text{(iii)}$$

The solution is completed in Table 16.1.

From Table 16.1

$$\Delta_{BD} = \frac{2.71PL}{AE} \qquad a_{BD} = \frac{4.82L}{AE}$$

Substituting these values in Eq. (i) we have

$$\frac{2.71PL}{AE} + X_{BD}\frac{4.82L}{AE} = 0$$

from which

$$X_{BD} = -0.56P \quad \text{(i.e. compression)}$$

The actual forces, $F_{a,j}$, in the members of the complete truss of Fig. 16.18(a) are now calculated using the method of joints and are listed in the final column of Table 16.1.

We note in the above that Δ_{BD} is positive, which means that Δ_{BD} is in the direction of the unit loads, i.e. B approaches D and the diagonal BD in the released structure decreases in length. Therefore in the complete structure the member BD, which prevents this shortening, must be in compression as shown; also a_{BD} will always be positive since it contains the term $F_{1,j}^2$. Finally, we note that the cut member BD is included in the calculation of the displacements in the released structure since its deformation, under a unit load, contributes to a_{BD}.

EXAMPLE 16.9

Calculate the forces in the members of the truss shown in Fig. 16.19(a). All members have the same cross sectional area, A, and Young's modulus, E.

By inspection we see that the truss is both internally and externally statically indeterminate since it would remain stable and in equilibrium if one of the diagonals, AD or BD, and the support at C were removed; the degree of indeterminacy is therefore 2. Unlike the truss in Fig. 16.18, we could not remove *any* member since, if BC or CD were removed, the outer half of the truss would become a mechanism while the portion ABDE would remain statically indeterminate. Therefore we select AD and the support at C as the releases, giving the statically determinate truss shown in Fig. 16.19 (b); we shall designate the force in the member AD as X_1 and the vertical reaction at C as R_2.

In this case we shall have two compatibility conditions, one for the diagonal AD and one for the support at C. We therefore need to investigate three loading cases: one in which the actual loads are applied to the released statically determinate truss in Fig. 16.19(b), a second in which unit loads are applied to the cut member AD (Fig. 16.19(c)) and a third in which a unit load is applied at C in the direction of R_2 (Fig. 16.19(d)). By comparison with the previous example, the compatibility conditions are

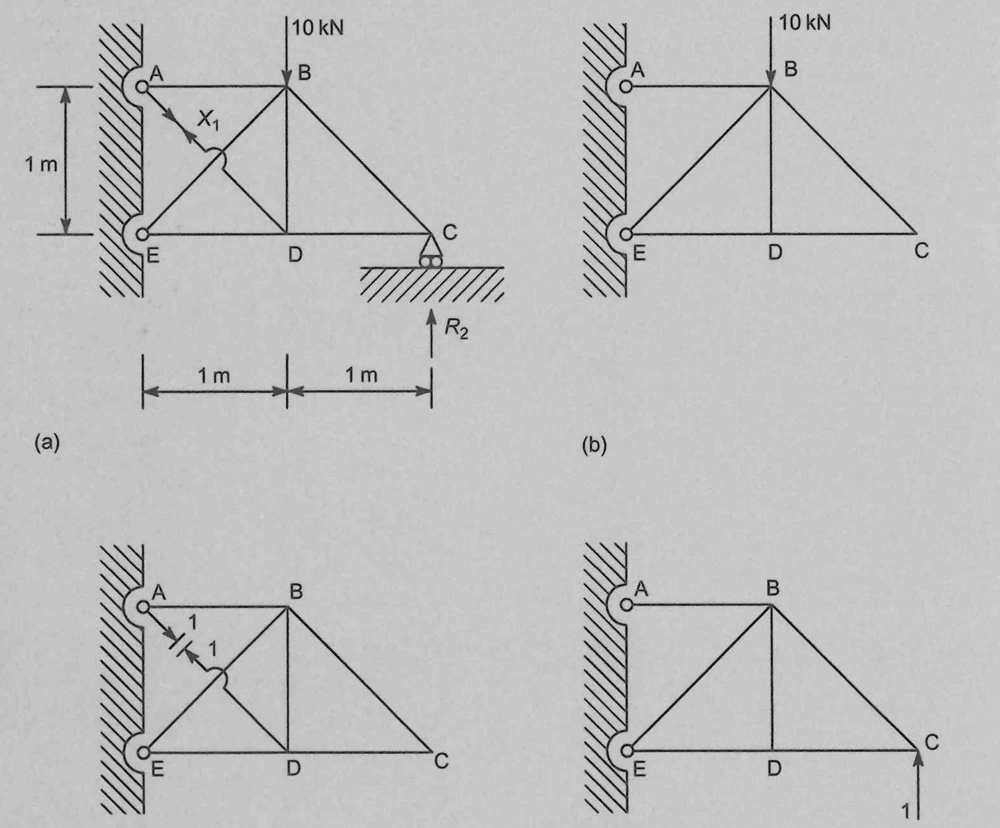


FIGURE 16.19

Table 16.2

Member	L_j	$F_{0,j}$	$F_{1,j}\,(X_1)$	$F_{1,j}\,(R_2)$	$F_{0,j}F_{1,j}$ $(X_1)L_j$	$F_{0,j}F_{1,j}$ $(R_2)L_j$	$F^2_{1,j}$ $(X_1)L_j$	$F^2_{1,j}$ $(R_2)L_j$	$F_{1,j}(X_1)$ $F_{1,j}(R_2)L_j$	$F_{a,j}$
AB	1	10.0	−0.71	−2.0	−7.1	−20.0	0.5	4.0	1.41	0.67
BC	1.41	0	0	−1.41	0	0	0	2.81	0	−4.45
CD	1	0	0	1.0	0	0	0	1.0	0	3.15
DE	1	0	−0.71	1.0	0	0	0.5	1.0	−0.71	0.12
AD	1.41	0	1.0	0	0	0	1.41	0	0	4.28
BE	1.41	−14.14	1.0	1.41	−20.0	−28.11	1.41	2.81	2.0	−5.4
BD	1	0	−0.71	0	0	0	0.5	0	0	−3.03
					$\Sigma = -27.1$	$\Sigma = -48.11$	$\Sigma = 4.32$	$\Sigma = 11.62$	$\Sigma = 2.7$	

$$\Delta_{AD} + a_{11}X_1 + a_{12}R_2 = 0 \qquad \text{(i)}$$

$$v_C + a_{21}X_1 + a_{22}R_2 = 0 \qquad \text{(ii)}$$

in which Δ_{AD} and v_C are, respectively, the change in length of the diagonal AD and the vertical displacement of C due to the actual loads acting on the released truss, while a_{11}, a_{12}, etc., are flexibility coefficients, which we have previously defined (see Ex. 16.7). The calculations are similar to those carried out in Ex. 16.8 and are shown in Table 16.2.

From Table 16.2

$$\Delta_{AD} = \sum_{j=1}^{n} \frac{F_{0,j}F_{1,j}(X_1)L_j}{AE} = \frac{-27.1}{AE} \quad \text{(i.e. AD increases in length)}$$

$$v_C = \sum_{j=1}^{n} \frac{F_{0,j}F_{1,j}(R_2)L_j}{AE} = \frac{-48.11}{AE} \quad \text{(i.e. C displaced downwards)}$$

$$a_{11} = \sum_{j=1}^{n} \frac{F^2_{1,j}(X_1)L_j}{AE} = \frac{4.32}{AE}$$

$$a_{22} = \sum_{j=1}^{n} \frac{F^2_{1,j}(R_2)L_j}{AE} = \frac{11.62}{AE}$$

$$a_{12} = a_{24} = \sum_{j=1}^{n} \frac{F_{1,j}(X_1)F_{1,j}(R_2)L_j}{AE} = \frac{2.7}{AE}$$

Substituting in Eqs (i) and (ii) and multiplying through by AE we have

$$-27.1 + 4.32X_1 + 2.7R_2 = 0 \qquad \text{(iii)}$$

$$-48.11 + 2.7X_1 + 11.62R_2 = 0 \qquad \text{(iv)}$$

Solving Eqs (iii) and (iv) we obtain

$$X_1 = 4.28 \text{ kN} \quad R_2 = 3.15 \text{ kN}$$

The actual forces, $F_{a,j}$, in the members of the complete truss are now calculated by the method of joints and are listed in the final column of Table 16.2.

Self-straining trusses

Statically indeterminate trusses, unlike the statically determinate type, may be subjected to self-straining in which internal forces are present before external loads are applied. Such a situation may be caused by a local temperature change or by an initial lack of fit of a member. In cases such as these, the term on the right-hand side of the compatibility equations, Eq. (ii) in Ex. 16.8 and Eqs (i) and (ii) in Ex. 16.9, would not be zero.

EXAMPLE 16.10

The truss shown in Fig. 16.20(a) is unstressed when the temperature of each member is the same, but due to local conditions the temperature in the member BC is increased by 30°C. If the cross-sectional area of each member is 200 mm^2 and the coefficient of linear expansion of the members is 7×10^{-6}/°C, calculate the resulting forces in the members; Young's modulus $E = 200\,000$ N/mm^2.

Due to the temperature rise, the increase in length of the member BC is $3 \times 10^3 \times 30 \times 7 \times 10^{-6} = 0.63$ mm. The truss has a degree of internal statical indeterminacy equal to 1 (by inspection). We therefore release the truss by cutting the member BC, which has experienced the temperature rise, as shown in Fig. 16.20(b); we shall suppose that the force in BC is X_1. Since there are no external loads on the truss, Δ_{BC} is zero and the compatibility condition becomes

$$a_{11}X_1 = -0.63 \text{ mm} \qquad \text{(i)}$$

in which, as before

$$a_{11} = \sum_{j=1}^{n} \frac{F_{1,j}^2 L_j}{AE}$$

Note that the extension of BC is negative since it is opposite in direction to X_1. The solution is now completed in Table 16.3. Hence

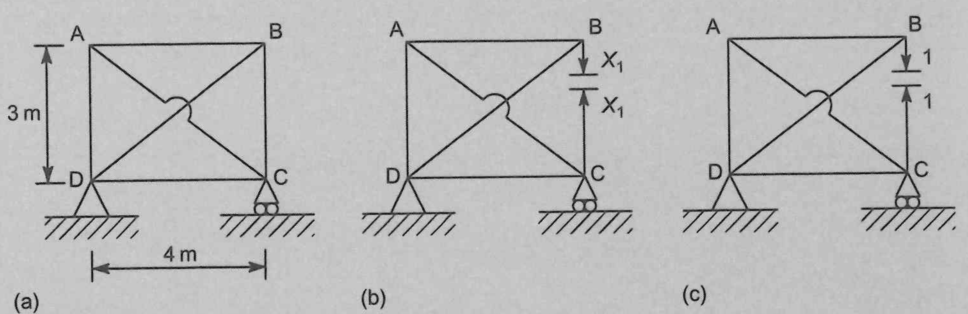


FIGURE 16.20

Self-straining due to a temperature change.

Table 16.3

Member	L_j(mm)	$F_{1,j}$	$F_{1,j}^2 L_j$	$F_{a,j}$ (N)
AB	4000	1.33	7111.1	−700
BC	3000	1.0	3000.0	−525
CD	4000	1.33	7111.1	−700
DA	3000	1.0	3000.0	−525
AC	5000	−1.67	13 888.9	875
DB	5000	−1.67	13 888.9	875
			$\Sigma = 48\,000.0$	

$$a_{11} = \frac{48\,000}{200 \times 200\,000} = 1.2 \times 10^{-3}$$

Thus, from Eq. (i)

$$X_1 = -525 \text{ N}$$

The forces, $F_{a,j}$, in the members of the complete truss are given in the final column of Table 16.3.

An alternative approach to the solution of statically indeterminate trusses, both self-straining and otherwise, is to use the principle of the stationary value of the total complementary energy.

Thus, for the truss of Ex. 16.8, the total complementary energy, C, is, from Eq. (15.39), given by

$$C = \sum_{j=1}^{n} \int_0^{F_j} \delta_j \, dF_j - P\Delta_C$$

in which Δ_C is the displacement of the joint C in the direction of P. Let us suppose that the member BD is short by an amount λ_{BD} (i.e. the lack of fit of BD), then

$$C = \sum_{j=1}^{n} \int_0^{F_j} \delta_j \, dF_j - P\Delta_C - X_1\lambda_{BD}$$

From the principle of the stationary value of the total complementary energy we have

$$\frac{\partial C}{\partial X_1} = \sum_{j=1}^{n} \delta_j \frac{\partial F_j}{\partial X_1} - \lambda_{BD} = 0 \qquad (16.8)$$

Assuming that the truss is linearly elastic, Eq. (16.8) may be written

$$\frac{\partial C}{\partial X_1} = \sum_{j=1}^{n} \frac{F_j L_j}{A_j E_j} \frac{\partial F_j}{\partial X_1} - \lambda_{BD} = 0 \qquad (16.9)$$

or since, for linearly elastic systems, the complementary energy, C, and the strain energy, U, are interchangeable,

$$\frac{\partial U}{\partial X_1} = \sum_{j=1}^{n} \frac{F_j L_j}{A_j E_j} \frac{\partial F_j}{\partial X_1} = \lambda_{BD} \qquad (16.10)$$

Equation (16.10) expresses mathematically what is generally referred to as Castigliano's second theorem which states that

For a linearly elastic structure the partial differential coefficient of the total strain energy of the structure with respect to the force in a redundant member is equal to the initial lack of fit of that member.

The application of complementary energy to the solution of statically indeterminate trusses is very similar to the method illustrated in Exs 16.8–16.10. For example, the solution of Ex. 16.8 would proceed as follows.

Again we select BD as the redundant member and suppose that the force in BD is X_1. The forces, $F_{a,j}$, in the complete truss are calculated in terms of P and X_1, and hence $\partial F_{a,j}/\partial X_1$ obtained for each

Table 16.4

Member	L_j	$F_{a,j}$	$\partial F_{a,j}/\partial X_1$	$F_{a,j}L_j\,(\partial F_{a,j}/\partial X_1)$
AB	L	$-0.71X_1$	-0.71	$0.5LX_1$
BC	L	$-0.71X_1$	-0.71	$0.5LX_1$
CD	L	$-P-0.71X_1$	-0.71	$(0.71P+0.5X_1)L$
DA	L	$-0.71X_1$	-0.71	$0.5LX_1$
AC	$1.41L$	$1.41P+X_1$	1.0	$(2P+1.41X_1)L$
BD	$1.41L$	X_1	1.0	$1.41X_1L$
				$\Sigma = 2.71PL + 4.82X_1L$

omplete truss. Equation (16.9) (or (16.10)) in which $\lambda_{BD} = 0$ then gives X_1 in terms of P. The soluon is illustrated in Table 16.4. Thus from Eq. (16.9)

$$\frac{1}{AE}(2.71PL + 4.82X_1L) = 0$$

om which

$$X_1 = -0.56P$$

s before.

Of the two approaches illustrated by the two solutions of Ex. 16.8, it can be seen that the use of the rinciple of the stationary value of the total complementary energy results in a slightly more algebraially clumsy solution. This will be even more the case when the degree of indeterminacy of a structure greater than 1 and the forces $F_{a,j}$ are expressed in terms of the applied loads and all the redundant orces. There will, of course, be as many equations of the form of Eq. (16.9) as there are redundancies.

6.6 Braced beams

ome structures consist of beams that are stiffened by trusses in which the beam portion of the structure is capable of resisting shear forces and bending moments in addition to axial forces. Generally, owever, displacements produced by shear forces are negligibly small and may be ignored. Therefore, n such structures we shall assume that the members of the truss portion of the structure resist axial orces only while the beam portion resists bending moments and axial forces; in some cases the axial orces in the beam are also ignored since their effect, due to the larger area of cross section, is small.

EXAMPLE 16.11

The beam ABC shown in Fig. 16.21(a) is simply supported and stiffened by a truss whose members are capable of resisting axial forces only. The beam has a cross-sectional area of 6000 mm^2 and a second moment of area of 7.2×10^6 mm^4. If the cross-sectional area of the members of the truss is 400 mm^2, calculate the forces in the members of the truss and the maximum value of the bending moment in the beam. Young's modulus, E, is the same for all members.

We observe that if the beam were capable of resisting axial forces only, the structure would be a relatively simple statically determinate truss. However, the beam, in addition to axial forces, resists

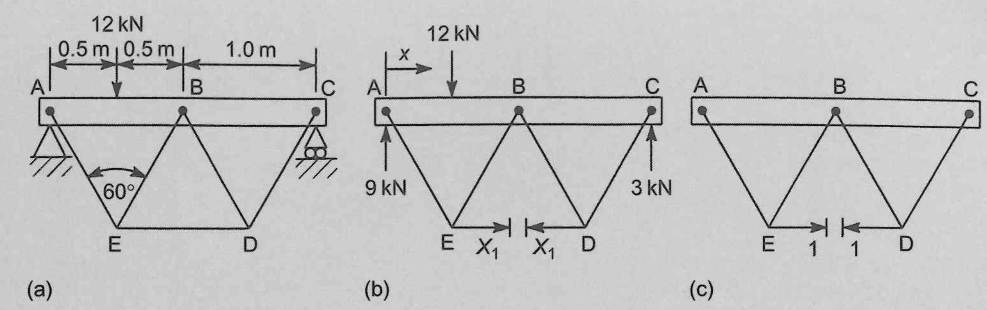


FIGURE 16.21

Braced beam of Ex. 16.11.

bending moments (we are ignoring the effect of shear) so that the structure is statically indeterminate with a degree of indeterminacy equal to 1, the bending moment at any section of the beam. Therefore we require just one release to produce a statically determinate structure; it does not necessarily have to be the bending moment in the beam, so we shall choose the truss member ED as shown in Fig. 16.21(b) since this will produce benefits from symmetry when we consider the unit load application in Fig. 16.21(c).

In this example displacements are produced by the bending of the beam as well as by the axial forces in the beam and truss members. Thus, in the released structure of Fig. 16.21(b), the relative displacement, Δ_{ED}, of the cut ends of the member ED is, from the unit load method (see Eq. (iii) of Ex. 15.13 and Exs 16.8–16.10), given by

$$\Delta_{ED} = \int_{ABC} \frac{M_0 M_1}{EI}\,dx + \sum_{j=1}^{n} \frac{F_{0,j}F_{1,j}L_j}{A_j E} \tag{i}$$

in which M_0 is the bending moment at any section of the beam ABC in the released structure. Further, the flexibility coefficient, a_{11}, of the member ED is given by

$$a_{11} = \int_{ABC} \frac{M_1^2}{EI}\,dx + \sum_{j=1}^{n} \frac{F_{1,j}^2 L_j}{A_j E} \tag{ii}$$

In Eqs (i) and (ii) the length, L_j, is constant, as is Young's modulus, E. These may therefore be omitted in the calculation of the summation terms in Table 16.5.

Examination of Table 16.5 shows that the displacement, Δ_{ED}, in the released structure is due solely to the bending of the beam, i.e. the second term on the right-hand side of Eq. (i) is zero; this could have been deduced by inspection of the released structure. Also the contribution to displacement of the axial forces in the beam may be seen, from the first two terms in the penultimate column of Table 16.5, to be negligibly small.

The contribution to Δ_{ED} of the bending of the beam will now be calculated. Thus from Fig. 16.21(b)

$$M_0 = 9x \quad (0 \le x \le 0.5 \text{ m})$$
$$M_0 = 9x - 12(x - 0.5) = 6 - 3x \quad (0.5 \le x \le 2.0 \text{ m})$$
$$M_1 = -0.87x \quad (0 \le x \le 1.0 \text{ m})$$

Table 16.5

Member	A_j (mm²)	$F_{0,j}$ (kN)	$F_{1,j}$	$F_{0,j}F_{1,j}/A_j$	$F_{1,j}^2/A_j$	$F_{a,j}$ (kN)
AB	6000	0	−0.5	0	4.17×10^{-5}	−2.01
BC	6000	0	−0.5	0	4.17×10^{-5}	−2.01
CD	400	0	1.0	0	2.5×10^{-3}	4.02
ED	400	0	1.0	0	2.5×10^{-3}	4.02
BD	400	0	−1.0	0	2.5×10^{-3}	−4.02
EB	400	0	−1.0	0	2.5×10^{-3}	−4.02
AE	400	0	1.0	0	2.5×10^{-3}	4.02
				$\Sigma=0$	$\Sigma=0.0126$	

Substituting from M_0 and M_1 in Eq. (i) we have

$$\int_{ABC}\frac{M_0M_1}{EI}\,dx$$

$$=\frac{1}{EI}\left[-\int_0^{0.5}9\times0.87x^2\,dx-\int_{0.5}^{1.0}(6-3x)0.87x\,dx+\int_{1.0}^{2.0}(6-3x)(0.87x-1.74)dx\right]$$

from which

$$\int_{ABC}\frac{M_0M_1}{EI}\,dx=-\frac{0.33\times10^6}{E}\,\text{mm}$$

Similarly

$$\int_{ABC}\frac{M_1^2}{EI}\,dx=\frac{1}{EI}\left[\int_0^{1.0}0.87^2x^2\,dx+\int_{1.0}^{2.0}(0.87x-1.74)^2\,dx\right]$$

from which

$$\int_{ABC}\frac{M_1^2}{EI}\,dx=\frac{0.083\times10^3}{EI}\,\text{mm/N}$$

The compatibility condition gives

$$\Delta_{ED}+a_{11}X_1=0$$

so that

$$-\frac{0.33\times10^6}{E}+\frac{0.083\times10^3}{E}X_1=0$$

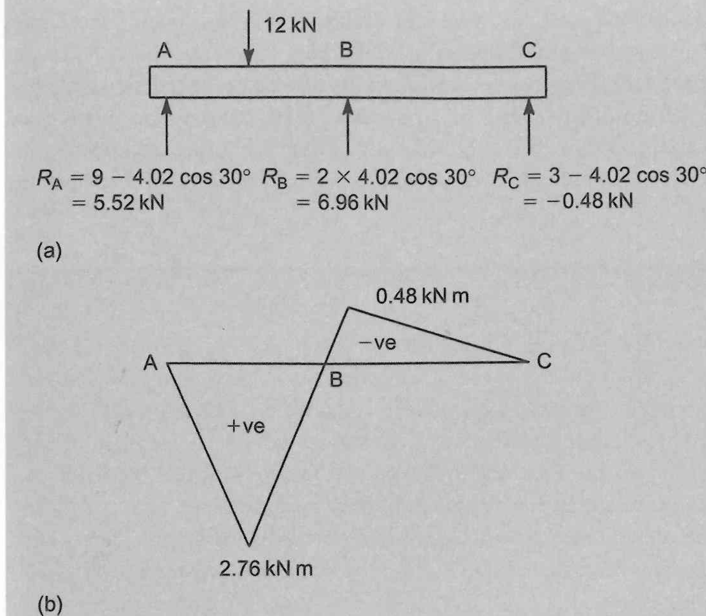


FIGURE 16.22

Bending moment distribution in the beam of Ex. 16.11.

which gives

$$X_1=4018.1\text{ N}\quad\text{or}\quad X_1=4.02\text{ kN}$$

The axial forces in the beam and truss may now be calculated using the method of joints and are given in the final column of Table 16.5. The forces acting on the beam in the complete structure are shown in Fig. 16.22(a) together with the bending moment diagram in Fig. 16.22(b), from which we see that the maximum bending moment in the beam is 2.76 kN m.

16.7 Portal frames

The flexibility method may be applied to the analysis of portal frames although, as we shall see, in all but simple cases the degree of statical indeterminacy is high so that the number of compatibility equations requiring solution becomes too large for hand computation.

Consider the portal frame shown in Fig. 16.23(a). From Section 16.2 we see that the frame, together with its foundation, form a single two-dimensional ring and is therefore three times statically indeterminate. Therefore we require 3 releases to obtain the statically determinate primary structure. These may be obtained by removing the foundation at the foot of one of the vertical legs as shown in Fig. 16.23(b); we

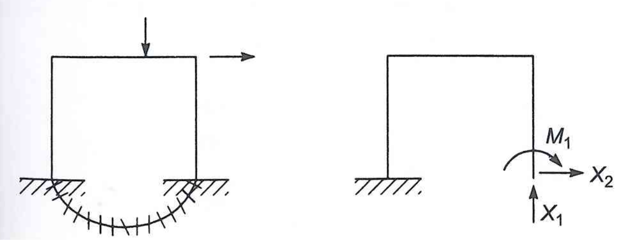


FIGURE 16.23

hen have two releases of force and one of moment and the primary structure is, in effect, a cranked cantiever. In this example there would be three compatibility equations requiring solution, two of translation and one of rotation. Clearly, for a plane, two-bay portal frame we would have six compatibility equations o that the solution would then become laborious; further additions to the frame would make a hand nethod of solution impracticable. Furthermore, as we shall see in Section 16.10, the moment distribution nethod produces a rapid solution for frames although it should be noted that using this method requires hat the sway of the frame, that is its lateral movement, is considered separately whereas, in the flexibility nethod, sway is automatically included.

EXAMPLE 16.12

Determine the distribution of bending moment in the frame ABCD shown in Fig. 16.24(a); the flexural rigidity of all the members of the frame is *EI*. Comparison with Fig. 16.23(a) shows that the frame has a degree of statical indeterminacy equal to 2 since the vertical leg CD is pinned to the foundation at D. We therefore require just 2 releases of reaction, as shown in Fig. 16.24(b), to obtain the statically determinate primary structure. For frames of this type it is usual to neglect the displacements produced by axial force and to assume that they are caused solely by bending.

The point D in the primary structure will suffer vertical and horizontal displacements, $\Delta_{D,V}$ and $\Delta_{D,H}$. Thus if we designate the redundant reactions as R_1, and R_2, the equations of compatibility are

$$\Delta_{D,V} + a_{11}R_1 + a_{12}R_2 = 0 \qquad \text{(i)}$$

$$\Delta_{D,H} + a_{21}R_1 + a_{22}R_2 = 0 \qquad \text{(ii)}$$

in which the flexibility coefficients have their usual meaning. Again, as in the preceding examples, we employ the unit load method to calculate the displacements and flexibility coefficients. Thus

$$\Delta_{D,V} = \sum \int_L \frac{M_0 M_{1,V}}{EI} dx$$

in which $M_{1,V}$ is the bending moment at any point in the frame due to a unit load applied vertically at D.

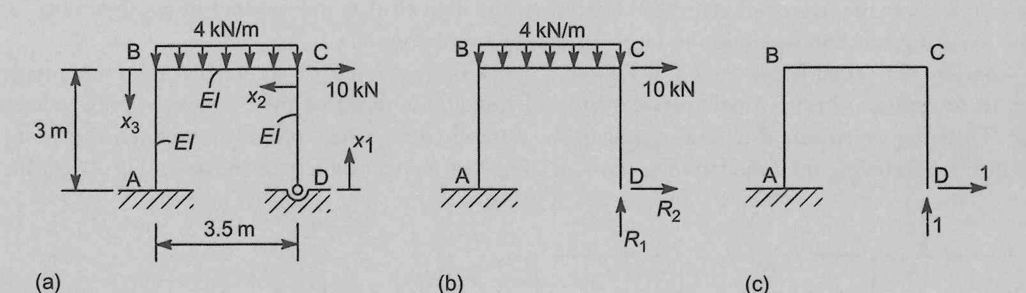


FIGURE 16.24

Portal frame of Ex. 16.12.

Similarly

$$\Delta_{D,H} = \sum \int_L \frac{M_0 M_{1,H}}{EI} dx$$

and

$$a_{11} = \sum \int_L \frac{M_{1,V}^2}{EI} dx \quad a_{22} = \sum \int_L \frac{M_{1,H}^2}{EI} dx \quad a_{12} = a_{21} = \sum \int_L \frac{M_{1,V} M_{1,H}}{EI} dx$$

We shall now write down expressions for bending moment in the members of the frame; we shall designate a bending moment as positive when it causes tension on the outside of the frame. Thus in DC

$$M_0 = 0 \quad M_{1,V} = 0 \quad M_{1,H} = -1x_1$$

In CB

$$M_0 = 4x_2 \frac{x_2}{2} = 2x_2^2 \quad M_{1,V} = -1x_2 \quad M_{1,H} = -3$$

In BA

$$M_0 = 4 \times 3.5 \times 1.75 + 10x_3 = 24.5 + 10x_3 \quad M_{1,V} = -3.5 \quad M_{1,H} = -1(3 - x_3)$$

Hence

$$\Delta_{D,V} = \frac{1}{EI}\left[\int_0^{3.5} (-2x_2^3) dx_2 + \int_0^3 -(24.5 + 10x_3)3.5\, dx_3\right] = -\frac{489.8}{EI}$$

$$\Delta_{D,H} = \frac{1}{EI}\left[\int_0^{3.5} (-6x_2^2) dx_2 + \int_0^3 -(24.5 + 10x_3)(3 - x_3) dx_3\right] = -\frac{241.0}{EI}$$

$$a_{11} = \frac{1}{EI}\left[\int_0^{3.5} x_2^2\, dx_2 + \int_0^3 3.5^2\, dx_3\right] = \frac{51.0}{EI}$$

$$a_{22} = \frac{1}{EI}\left[\int_0^3 x_1^2\, dx_1 + \int_0^{3.5} 3^2\, dx_2 + \int_0^3 (3 - x_3)^2\, dx_3\right] = \frac{49.5}{EI}$$

$$a_{12} = a_{21} = \frac{1}{EI}\left[\int_0^{3.5} 3x_2\, dx_2 + \int_0^3 3.5(3 - x_3) dx_3\right] = \frac{34.1}{EI}$$

Substituting for $\Delta_{D,V}$, $\Delta_{D,H}$, a_{11}, etc., in Eqs (i) and (ii) we obtain

$$-\frac{489.8}{EI} + \frac{51.0}{EI}R_1 + \frac{34.1}{EI}R_2 = 0 \qquad \text{(iii)}$$

and

$$-\frac{241.0}{EI} + \frac{34.1}{EI}R_1 + \frac{49.5}{EI}R_2 = 0 \qquad \text{(iv)}$$

Solving Eqs (iii) and (iv) we have

$$R_1 = 11.8 \text{ kN} \quad R_2 = -3.3 \text{ kN}$$

The bending moment diagram is then drawn as shown in Fig. 16.25.

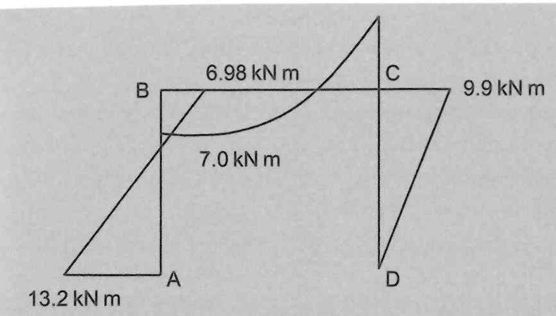


FIGURE 16.25

Bending moment diagram for the frame of Ex. 16.12 (diagram drawn on tension side of members).

It can be seen that the amount of computation for even the relatively simple frame of Ex. 16.12 is quite considerable. Generally, therefore, as stated previously, the moment distribution method or a computer-based analysis would be employed.

16.8 Two-pinned arches

In Chapter 6 we saw that a three-pinned arch is statically determinate due to the presence of the third pin or hinge at which the internal bending moment is zero; in effect the presence of the third pin provides a release. Therefore a two-pinned arch such as that shown in Fig. 16.26(a) has a degree of statical indeterminacy equal to 1. This is also obvious from inspection since, as in the three-pinned arch, there are two reactions at each of the supports.

The analysis of two-pinned arches, i.e. the determination of the support reactions, may be carried out using the flexibility method; again, as in the case of portal frames, it is usual to ignore the effect of axial force on displacements and to assume that they are caused by bending action only.

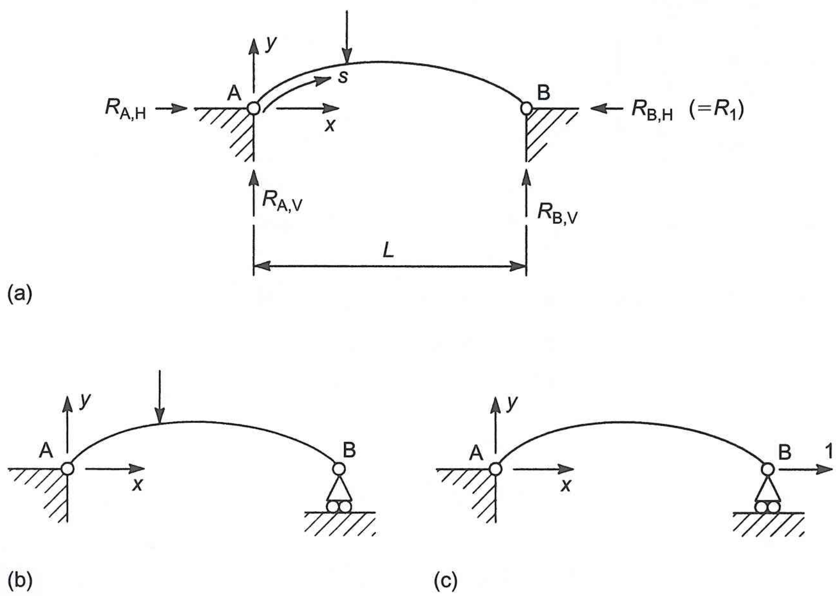


FIGURE 16.26

The arch in Fig. 16.26(a) has a profile whose equation may be expressed in terms of the reference axes x and y. The second moment of area of the cross section of the arch is I and we shall designate the distance round the profile from A as s.

Initially we choose a release, say the horizontal reaction, R_1, at B, to obtain the statically determinate primary structure shown in Fig. 16.26(b). We then employ the unit load method to determine the horizontal displacement, $\Delta_{B,H}$, of B in the primary structure and the flexibility coefficient, a_{11}. Then, from compatibility

$$\Delta_{B,H} - a_{11}R_1 = 0 \tag{16.11}$$

in which the term containing R_1 is negative since R_1 is opposite in direction to the unit load (see Fig. 16.26(c)).

Then, with the usual notation

$$\Delta_{B,H} = \int_{\text{Profile}} \frac{M_0 M_1}{EI}\, ds \tag{16.12}$$

in which M_0 depends upon the applied loading and $M_1 = 1y$ (a moment is positive if it produces tension on the undersurface of the arch). Also

$$a_{11} = \int_{\text{Profile}} \frac{M_1^2}{EI}\, ds = \int_{\text{Profile}} \frac{y^2}{EI}\, ds \tag{16.13}$$

Substituting for M_1 in Eq. (16.12) and then for $\Delta_{B,H}$ and a_{11} in Eq. (16.11) we obtain

$$R_1 = \frac{\int_{\text{Profile}}(M_0 y/EI)\, ds}{\int_{\text{Profile}}(y^2/EI)\, ds} \tag{16.14}$$

EXAMPLE 16.13

Determine the support reactions in the semicircular two-pinned arch shown in Fig. 16.27(a). The flexural rigidity, EI, of the arch is constant throughout.

Again we shall choose the horizontal reaction at the support B as the release so that $R_{B,H}$ $(=R_1)$ is given directly by Eq. (16.14) in which M_0 and s are functions of x and y. The computation will therefore be simplified if we use an angular coordinate system so that, from the primary structure shown in Fig. 16.27(b)

$$M_0 = R'_{B,V}(5 + 5\cos\theta) - \frac{10}{2}(5 + 5\cos\theta)^2 \tag{i}$$

in which $R'_{B,V}$ is the vertical reaction at B in the primary structure. From Fig. 16.27(b) in which, from symmetry, $R'_{B,V} = R'_{A,V}$, we have $R'_{B,V} = 50$ kN. Substituting for $R'_{B,V}$ in Eq. (i) we obtain

$$M_0 = 125\sin^2\theta \tag{ii}$$

Also $y = 5\sin\theta$ and $ds = 5\, d\theta$, so that from Eq. (16.14) we have

$$R_1 = \frac{\int_0^{\pi} 125\sin^2\theta\, 5\sin\theta\, 5\, d\theta}{\int_0^{\pi} 25\sin^2\theta\, 5\, d\theta}$$

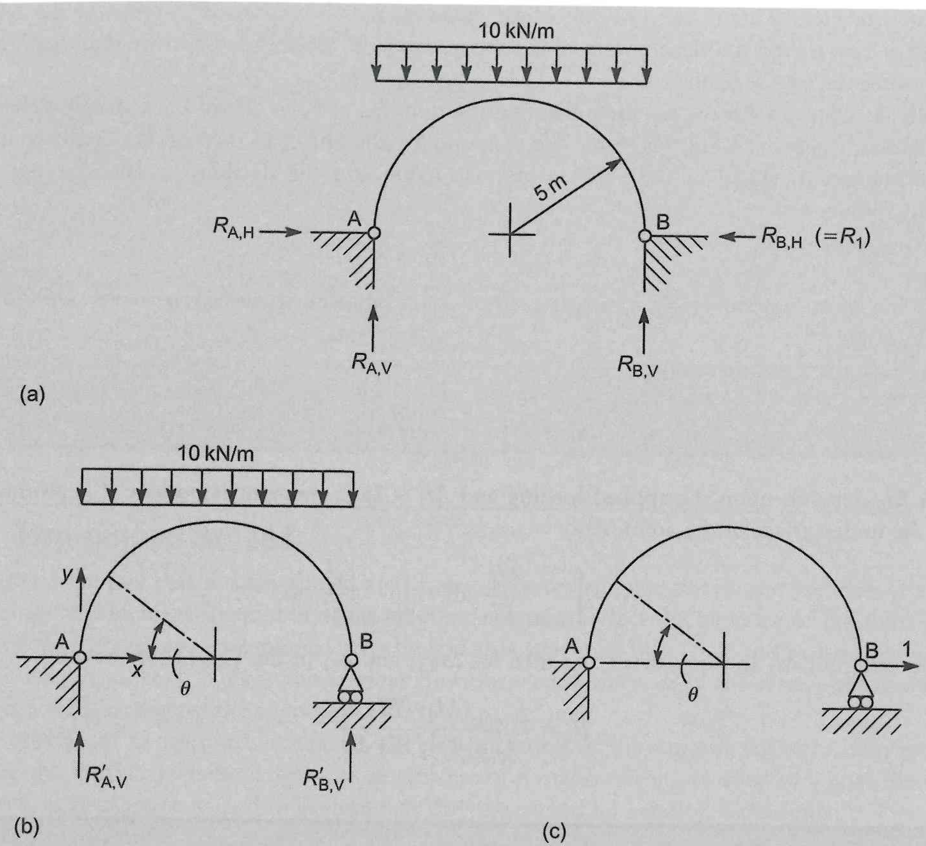


FIGURE 16.27

Semicircular arch of Ex. 16.13.

or

$$R_1 = \frac{\int_0^{\pi} 25 \sin^3 \theta \, d\theta}{\int_0^{\pi} \sin^2 \theta \, d\theta} \tag{iii}$$

which gives

$$R_1 = 21.2 \text{ kN}(=R_{B,H})$$

The remaining reactions follow from a consideration of the statical equilibrium of the arch and are

$$R_{A,H} = 21.2 \text{ kN} \quad R_{A,V} = R_{B,V} = 50 \text{ kN}$$

The integrals in Eq. (iii) of Ex. 16.13 are relatively straightforward to evaluate; the numerator may be found by integration by parts, while the denominator is found by replacing $\sin^2 \theta$ by $(1 - \cos 2\theta)/2$. Furthermore, in an arch having a semicircular profile, M_0, y and ds are simply expressed in terms of an angular coordinate system. However, in a two-pinned arch having a parabolic profile this approach cannot be used and complex integrals result. Such cases may be simplified by specifying that the second moment of area of the cross section of the arch varies round the profile; one such variation is known as the secant assumption and is described below.

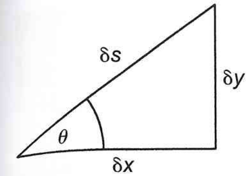


FIGURE 16.28

Elemental length of arch.

Secant assumption

In Eq. (16.14) the term ds/I appears. If this term could be replaced by a term that is a function of either x or y, the solution would be simplified.

Consider the elemental length, δs, of the arch shown in Fig. 16.28 and its projections, δx and δy, on the x and y axes. From the elemental triangle

$$\delta x = \delta s \cos \theta$$

or, in the limit as $\delta s \to 0$

$$ds = \frac{dx}{\cos \theta} = dx \sec \theta$$

Thus

$$\frac{ds}{I} = \frac{dx \sec \theta}{I}$$

Let us suppose that I varies round the profile of the arch such that $I = I_0 \sec \theta$ where I_0 is the second moment of area at the crown of the arch (i.e. where $\theta = 0$). Then

$$\frac{ds}{I} = \frac{dx \sec \theta}{I_0 \sec \theta} = \frac{dx}{I_0}$$

Thus substituting in Eq. (16.14) for ds/I we have

$$R_1 = \frac{\int_{\text{Profile}} (M_0 y / E I_0) dx}{\int_{\text{Profile}} (y^2 / E I_0) dx}$$

or

$$R_1 = \frac{\int_{\text{Profile}} M_0 y \, dx}{\int_{\text{Profile}} y^2 \, dx} \tag{16.15}$$

EXAMPLE 16.14

Determine the support reactions in the parabolic arch shown in Fig. 16.29 assuming that the second moment of area of the cross section of the arch varies in accordance with the secant assumption.

The equation of the arch may be shown to be

$$y = \frac{4h}{L^2}(Lx - x^2) \tag{i}$$

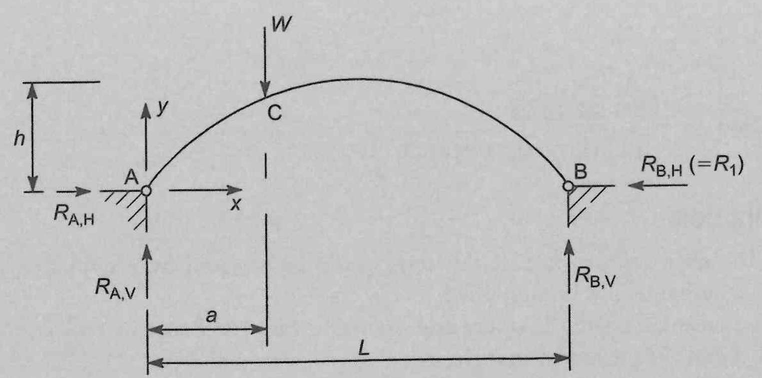


FIGURE 16.29

Parabolic arch of Ex. 16.14.

Again we shall release the arch at B as in Fig. 16.26(b). Then

$$M_0 = R'_{A,V}x \ (0 \leq x \leq a)$$
$$M_0 = R'_{A,V}x - W(x-a) \ (a \leq x \leq L)$$

in which $R'_{A,V}$ is the vertical reaction at A in the released structure. Now taking moments about B we have

$$R'_{A,V}L - W(L-a) = 0$$

from which

$$R'_{A,V} = \frac{W}{L}(L-a)$$

Substituting in the expressions for M_0 gives

$$M_0 = \frac{W}{L}(L-a)x \quad (0 \leq x \leq a) \qquad \text{(ii)}$$

$$M_0 = \frac{W}{L}(L-x) \quad (a \leq x \leq L) \qquad \text{(iii)}$$

The denominator in Eq. (16.15) may be evaluated separately. Thus, from Eq. (i)

$$\int_{\text{profile}} y^2 \mathrm{d}x = \int_0^L \left(\frac{4h}{L^2}\right)^2 (Lx - x^2)^2 \mathrm{d}x = \frac{8h^2L}{15}$$

Then, from Eq. (16.15) and Eqs (ii) and (iii)

$$R_1 = \frac{15}{8h^2L}\left[\int_0^a \frac{W}{L}(L-a)x\frac{4h}{L^2}(Lx-x^2)\mathrm{d}x + \int_a^L \frac{Wa}{L}(L-x)\frac{4h}{L^2}(Lx-x^2)\mathrm{d}x\right]$$

which gives

$$R_1 = \frac{5Wa}{8hL^3}(L^3 + a^3 - 2La^2) \qquad \text{(iv)}$$

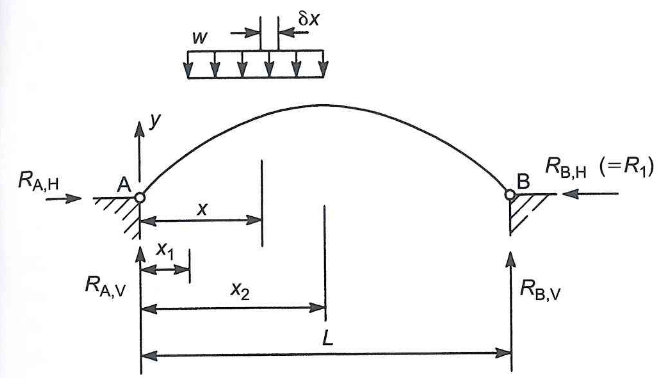


FIGURE 16.30

Parabolic arch carrying a part-span uniformly distributed load.

The remaining support reactions follow from a consideration of the statical equilibrium of the arch.

If, in Ex. 16.14, we had expressed the load position in terms of the span of the arch, say $a = kL$, Eq. (iv) in Ex. 16.14 becomes

$$R_1 = \frac{5WL}{8h}(k + k^4 - 2k^3) \qquad (16.16)$$

Therefore, for a series of concentrated loads positioned at distances k_1L, k_2L, k_3L, etc., from A, the reaction, R_1, may be calculated for each load acting separately using Eq. (16.16) and the total reaction due to all the loads obtained by superposition.

The result expressed in Eq. (16.16) may be used to determine the reaction, R_1, due to a part-span uniformly distributed load. Consider the arch shown in Fig. 16.30. The arch profile is parabolic and its second moment of area varies as the secant assumption. An elemental length, δx, of the load produces a load $w\,\delta x$ on the arch. Thus, since δx is very small, we may regard this load as a concentrated load. This will then produce an increment, δR_1, in the horizontal support reaction which, from Eq. (16.16), is given by

$$\delta R_1 = \frac{5}{8}w\delta x\frac{L}{h}(k + k^4 - 2k^3)$$

in which $k = x/L$. Therefore, substituting for k in the expression for δR_1 and then integrating over the length of the load we obtain

$$R_1 = \frac{5wL}{8h}\int_{x_1}^{x_2}\left(\frac{x}{L} + \frac{x^4}{L^4} - \frac{2x^3}{L^3}\right)\mathrm{d}x$$

which gives

$$R_1 = \frac{5wL}{8h}\left[\frac{x^2}{2L} + \frac{x^5}{5L^4} - \frac{x^4}{2L^3}\right]_{x_1}^{x_2}$$

For a uniformly distributed load covering the complete span, i.e. $x_1 = 0$, $x_2 = L$, we have

$$R_1 = \frac{5wL}{8h}\left(\frac{L^2}{2L} + \frac{L^5}{5L^4} - \frac{L^4}{2L^3}\right) = \frac{wL^2}{8h}$$

The bending moment at any point (x, y) in the arch is then

$$M = \frac{wL}{2}x - \frac{wx^2}{2} = -\frac{wL^2}{8h}\left[\frac{4h}{L^2}(Lx - x^2)\right]$$

i.e.

$$M = \frac{wL}{2}x - \frac{wx^2}{2} - \frac{wL}{2}x + \frac{wx^2}{2} = 0$$

Therefore, for a parabolic two-pinned arch carrying a uniformity distributed load over its complete span, the bending moment in the arch is everywhere zero; the same result was obtained for the three-pinned arch in Chapter 6.

Although the secant assumption appears to be an artificial simplification in the solution of parabolic arches it would not, in fact, produce a great variation in second moment of area in, say, large-span shallow arches. The assumption would therefore provide reasonably accurate solutions for some practical cases.

Tied arches

In some cases the horizontal support reactions are replaced by a tie which connects the ends of the arch as shown in Fig. 16.31(a). In this case we select the axial force, X_1, in the tie as the release. The primary structure is then as shown in Fig. 16.31(b) with the tie cut. The unit load method, Fig. 16.31(c), is then used to determine the horizontal displacement of B in the primary structure. This displacement will receive contributions from the bending of the arch and the axial force in the tie. Thus, with the usual notation

$$\Delta_{B,H} = \int_{Profile} \frac{M_0 M_1}{EI} ds + \int_0^L \frac{F_0 F_1 L}{AE} dx$$

and

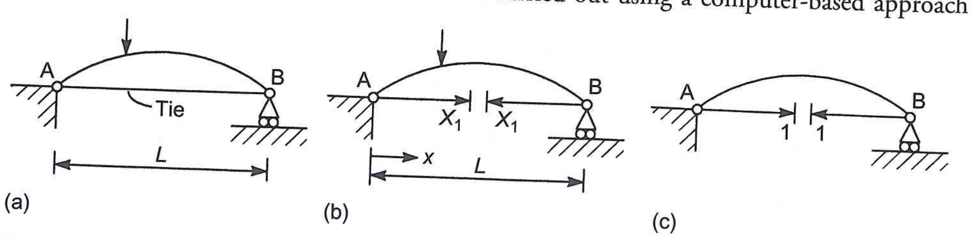
$$a_{11} = \int_{Profile} \frac{M_1^2}{EI} ds + \int_0^L \frac{F_1^2 L}{AE} dx$$

The compatibility condition is then

$$\Delta_{B,H} + a_{11} X_1 = 0$$

Segmental arches

A segmental arch is one comprising segments having different curvatures or different equations describ-ng their profiles. The analysis of such arches is best carried out using a computer-based approach such



IGURE 16.31

olution for a tied two-pinned arch.

as the stiffness method in which the stiffness of an individual segment may be found by determining the force–displacement relationships using an energy approach. Such considerations are, however, outside the scope of this book.

16.9 Slope–deflection method

An essential part of the computer-based stiffness method of analysis and also of the moment distribution method is the slope–deflection relationships for beam elements. In these, the shear forces and moments at the ends of a beam element are related to the end displacements and rotations. In addition these relationships provide a method of solution for the determination of end moments in statically indeterminate beams and frames; this method is known as the *slope–deflection method.*

Consider the beam, AB, of length L, shown in Fig. 16.32. The beam has a flexural rigidity EI and is subjected to moments, M_{AB} and M_{BA}, and shear forces, S_{AB} and S_{BA}, at its ends. The shear forces and moments produce displacements v_A and v_B and rotations θ_A and θ_B as shown. Here we are concerned with moments at the ends of a beam. The usual sagging/hogging sign convention is therefore insufficient to describe these moments since a clockwise moment at the left-hand end of a beam coupled with an anticlockwise moment at the right-hand end would induce a positive bending moment at all sections of the beam. We shall therefore adopt a sign convention such that the moment at a point is positive when it is applied in a clockwise sense and negative when in an anticlockwise sense; thus in Fig. 16.32 both moments M_{AB} and M_{BA} are positive. We shall see in the solution of a particular problem how these end moments are interpreted in terms of the bending moment distribution along the length of a beam. In the analysis we shall ignore axial force effects since these would have a negligible effect in the equation for moment equilibrium. Also, the moments M_{AB} and M_{BA} are independent of each other but the shear forces, which in the absence of lateral loads are equal and opposite, depend upon the end moments.

From Eq. (13.3) and Fig. 16.32

$$EI\frac{d^2v}{dx^2} = M_{AB} + S_{AB}x$$

Note that the actual displacements of the beam are small so that

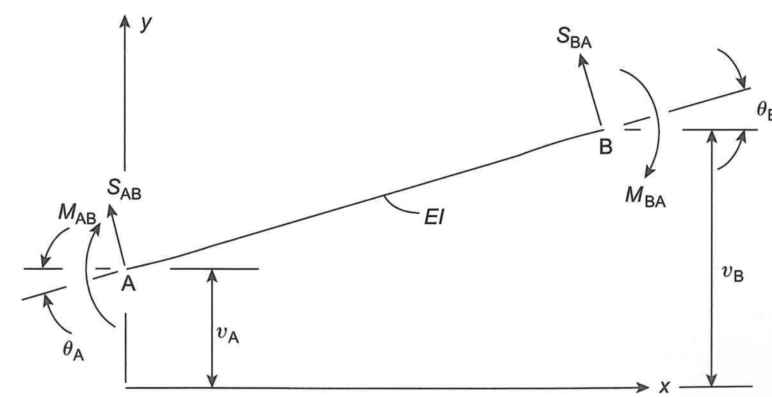


FIGURE 16.32

$$S_{AB}(x/\cos\theta) \approx S_{AB}\,x$$

Then

$$EI\frac{dv}{dx} = M_{AB}x + S_{AB}\frac{x^2}{2} + C_1 \qquad (16.17)$$

nd

$$EIv = M_{AB}\frac{x^2}{2} + S_{AB}\frac{x^3}{6} + C_1x + C_2 \qquad (16.18)$$

hen

$$x = 0 \quad \frac{dv}{dx} = \theta_A \quad v = v_A$$

Therefore, from Eq. (16.17) $C_1 = EI\theta_A$ and from Eq. (16.18), $C_2 = EIv_A$. Equations (16.17) and (16.18) then, respectively, become

$$EI\frac{dv}{dx} = M_{AB}x + S_{AB}\frac{x^2}{2} + EI\theta_A \qquad (16.19)$$

nd

$$EI_v = M_{AB}\frac{x^2}{2} + S_{AB}\frac{x^3}{6} + EI\theta_A x + EIv_A \qquad (16.20)$$

Also, at $x = L$, $dv/dx = \theta_B$ and $v = v_B$. Thus, from Eqs (16.19) and (16.20) we have

$$EI\theta_B = M_{AB}L + S_{AB}\frac{L^2}{2}EI\theta_A \qquad (16.21)$$

nd

$$EIv_B = M_{AB}\frac{L^2}{2} + S_{AB}\frac{L^3}{6} + EI\theta_A L + EIv_A \qquad 16.22)$$

Solving Eqs (16.21) and (16.22) for M_{AB} and S_{AB} gives

$$M_{AB} = -\frac{2EI}{L}\left[2\theta_A + \theta_B + \frac{3}{L}(v_A - v_B)\right] \qquad (16.23)$$

nd

$$S_{AB} = \frac{6EI}{L^2}\left[\theta_A + \theta_B + \frac{2}{L}(v_A - v_B)\right] \qquad (16.24)$$

Now, from the moment equilibrium of the beam about B, we have

$$M_{BA} + S_{AB}L + M_{AB} = 0$$

)

$$M_{BA} = -S_{AB}L - M_{AB}$$

Substituting for S_{AB} and M_{AB} in this expression from Eqs (16.24) and (16.23) we obtain

$$M_{BA} = -\frac{2EI}{L}\left[2\theta_B + \theta_A + \frac{3}{L}(v_A - v_B)\right] \qquad (16.25)$$

Further, Since $S_{BA} = -S_{AB}$ (from the vertical equilibrium of the element)

$$S_{BA} = -\frac{6EI}{L^2}\left[\theta_A + \theta_B + \frac{2}{L}(v_A - v_B)\right] \qquad (16.26)$$

Equations (16.23)–(16.26) are usually written in the form

$$\left.\begin{aligned} M_{AB} &= -\frac{6EI}{L^2}v_A - \frac{4EI}{L}\theta_A + \frac{6EI}{L^2}v_B - \frac{2EI}{L}\theta_B \\ S_{AB} &= \frac{12EI}{L^3}v_A + \frac{6EI}{L^2}\theta_A - \frac{12EI}{L^3}v_B + \frac{6EI}{L^2}\theta_B \\ M_{AB} &= -\frac{6EI}{L^3}v_A - \frac{2EI}{L}\theta_A + \frac{6EI}{L^3}v_B - \frac{4EI}{L}\theta_B \\ S_{BA} &= -\frac{12EI}{L^3}v_A - \frac{6EI}{L^2}\theta_A + \frac{12EI}{L^3}v_B - \frac{6EI}{L^2}\theta_B \end{aligned}\right\} \qquad (16.27)$$

Equation (16.27) are known as the slope–deflection equations and establish force–displacement relationships for the beam as opposed to the displacement–force relationships of the flexibility method. The coefficients that pre-multiply the components of displacement in Eq. (16.27) are known as *stiffness coefficients.*

The beam in Fig. 16.32 is not subject to lateral loads. Clearly, in practical cases, unless we are interested solely in the effect of a sinking support, lateral loads will be present. These will cause additional moments and shear forces at the ends of the beam. Equations (16.23)–(16.26) may then be written as

$$M_{AB} = -\frac{2EI}{L}\left[2\theta_A + \theta_B + \frac{3}{L}(v_A - v_B)\right] + M^F_{AB} \qquad (16.28)$$

$$S_{AB} = \frac{6EI}{L^2}\left[\theta_A + \theta_B + \frac{2}{L}(v_A - v_B)\right] + S^F_{AB} \qquad (16.29)$$

$$M_{BA} = -\frac{2EI}{L}\left[2\theta_B + \theta_A + \frac{3}{L}(v_A - v_B)\right] + M^F_{BA} \qquad (16.30)$$

$$S_{BA} = -\frac{6EI}{L^2}\left[\theta_A + \theta_B + \frac{2}{L}(v_A - v_B)\right] + S^F_{BA} \qquad (16.31)$$

in which M^F_{AB} and M^F_{BA} are the moments at the ends of the beam caused by the applied loads and correspond to $\theta_A = \theta_B = 0$ and $v_A = v_B = 0$, i.e. they are *fixed-end moments* (FEMs). Similarly the shear forces S^F_{AB} and S^F_{BA} correspond to the fixed-end case.

EXAMPLE 16.15

Find the support reactions in the three-span continuous beam shown in Fig. 16.33.

The beam in Fig. 16.33 is the beam that was solved using the flexibility method in Ex. 16.7, so that this example provides a comparison between the two methods.

Initially we consider the beam as comprising three separate fixed beams AB, BC and CD and calculate the values of the FEMs, M^F_{AB}, M^F_{BA}, M^F_{BC}, etc. Thus, using the results of Exs 13.22 and 13.24 and remembering that clockwise moments are positive and anticlockwise moments negative

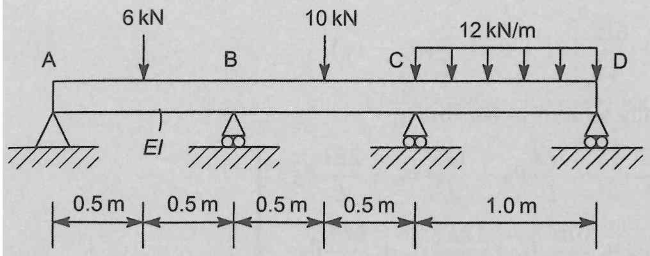


FIGURE 16.33
Continuous beam of Ex. 16.15.

$$M^F_{AB} = -M^F_{BA} = -\frac{6 \times 1.0}{8} = -0.75 \text{ kN m}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{10 \times 1.0}{8} = -1.25 \text{ kN m}$$

$$M^F_{CD} = -M^F_{DC} = -\frac{12 \times 1.0^2}{12} = -1.0 \text{ kN m}$$

In the beam of Fig. 16.33 the vertical displacements at all the supports are zero, i.e. v_A, v_B, v_C nd v_D are zero. Therefore, from Eqs (16.28) and (16.30) we have

$$M_{AB} = -\frac{2EI}{1.0}(2\theta_A + \theta_B) - 0.75 \quad \text{(i)}$$

$$M_{BA} = -\frac{2EI}{1.0}(2\theta_B + \theta_A) - 0.75 \quad \text{(ii)}$$

$$M_{BC} = -\frac{2EI}{1.0}(2\theta_B + \theta_C) - 1.25 \quad \text{(iii)}$$

$$M_{CB} = -\frac{2EI}{1.0}(2\theta_C + \theta_B) + 1.25 \quad \text{(iv)}$$

$$M_{CD} = -\frac{2EI}{1.0}(2\theta_C + \theta_D) - 1.0 \quad \text{(v)}$$

$$M_{DC} = -\frac{2EI}{1.0}(2\theta_D + \theta_C) + 1.0 \quad \text{(vi)}$$

From the equilibrium of moments at the supports

$$M_{AB} = 0 \quad M_{BA} + M_{BC} = 0 \quad M_{CB} + M_{CD} = 0 \quad M_{DC} = 0$$

Substituting for M_{AB}, etc., from Eqs (i)–(vi) in these expressions we obtain

$$4EI\theta_A + 2EI\theta_B + 0.75 = 0 \quad \text{(vii)}$$

$$2EI\theta_A + 8EI\theta_B + 2EI\theta_C + 0.5 = 0 \quad \text{(viii)}$$

$$2EI\theta_B + 8EI\theta_C + 2EI\theta_D - 0.25 = 0 \quad \text{(ix)}$$

$$4EI\theta_D + 2EI\theta_C - 1.0 = 0 \quad \text{(x)}$$

The solution of Eqs (vii)–(x) gives

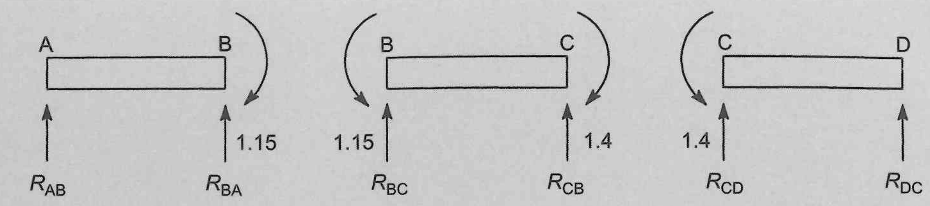


FIGURE 16.34
Moments and reactions at the ends of the spans of the continuous beam of Ex. 16.15.

Substituting these values in Eqs (i)–(vi) gives

$$M_{AB} = 0 \quad M_{BA} = 1.15 \quad M_{BC} = -1.15 \quad M_{CB} = 1.4 \quad M_{CD} = -1.4 \quad M_{DC} = 0$$

The end moments acting on the three spans of the beam are now shown in Fig. 16.34. They produce reactions R_{AB}, R_{BA}, etc., at the supports; thus

$$R_{AB} = -R_{BA} = -\frac{1.15}{1.0} = -1.15 \text{ kN}$$

$$R_{BC} = -R_{CB} = -\frac{(1.4 - 1.15)}{1.0} = -0.25 \text{ kN}$$

$$R_{CD} = -R_{DC} = \frac{1.4}{1.0} = 1.40 \text{ kN}$$

Therefore, due to the end moments *only*, the support reactions are

$$R_{A,M} = -1.15 \text{ kN} \quad R_{B,M} = 1.15 - 0.25 = 0.9 \text{ kN},$$

$$R_{C,M} = 0.25 + 1.4 = 1.65 \text{ kN} \quad R_{D,M} = -1.4 \text{ kN}$$

In addition to these reactions there are the reactions due to the actual loading, which may be obtained by analysing each span as a simply supported beam (the effects of the end moments have been calculated above). In this example these reactions may be obtained by inspection. Thus

$$R_{A,S} = 3.0 \text{ kN} \quad R_{B,S} = 3.0 + 5.0 = 8.0 \text{ kN} \quad R_{C,S} = 5.0 + 6.0 = 11.0 \text{ kN}$$

$$R_{D,S} = 6.0 \text{ kN}$$

The final reactions at the supports are then

$$R_A = R_{A,M} + R_{A,S} = -1.15 + 3.0 = 1.85 \text{ kN}$$

$$R_B = R_{B,M} + R_{B,S} = 0.9 + 8.0 = 8.9 \text{ kN}$$

$$R_C = R_{C,M} + R_{C,S} = 1.65 + 11.0 = 12.65 \text{ kN}$$

$$R_D = R_{D,M} + R_{D,S} = -1.4 + 6.0 = 4.6 \text{ kN}$$

Alternatively, we could have obtained these reactions by the slightly lengthier procedure of substituting for θ_A, θ_B, etc., in Eqs (16.29) and (16.31). Thus, e.g.

$$S_{AB} = R_A = \frac{6EI}{L^2}(\theta_A + \theta_B) + 3.0 \quad (v_A = v_B = 0)$$

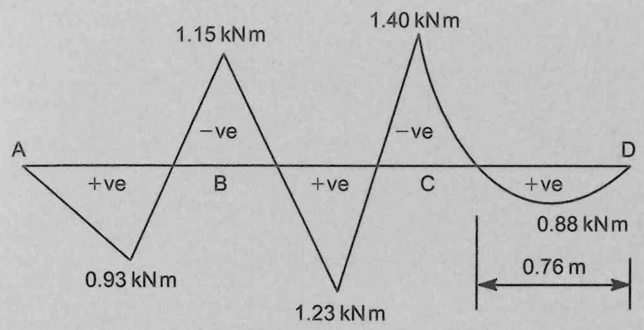


FIGURE 16.35
Bending moment diagram for the beam of Ex. 16.15.

Comparing the above solution with that of Ex. 16.7 we see that there are small discrepancies; :hese are caused by rounding-off errors.

Having obtained the support reactions, the bending moment distribution (reverting to the sag- ;ing (positive) and hogging (negative) sign convention) is obtained in the usual way and is shown in ⁻ig. 16.35.

EXAMPLE 16.16

Determine the end moments in the members of the portal frame shown in Fig. 16.36; the second noment of area of the vertical members is $2.5I$ while that of the horizontal members is I.

In this particular problem the approach is very similar to that for the continuous beam of Ex. 16.15. However, due to the unsymmetrical geometry of the frame and also to the application of the 10 kN load, the frame will sway such that there will be horizontal displacements, v_B and v_C, at B ınd C in the members BA and CD. Since we are ignoring displacements produced by axial forces :hen $v_B = v_C = v_1$, say. We would, in fact, have a similar situation in a continuous beam if one or nore of the supports experienced settlement. Also we note that the rotation, θ_A, at A must be zero ;ince the end A of the member AB is fixed.

Initially, as in Ex. 16.15, we calculate the FEMs in the members of the frame, again using the esults of Exs 13.22 and 13.24. The effect of the cantilever CE may be included by replacing it by its ɛnd moment, thereby reducing the number of equations to be solved. Thus, from Fig. 16.36 we have

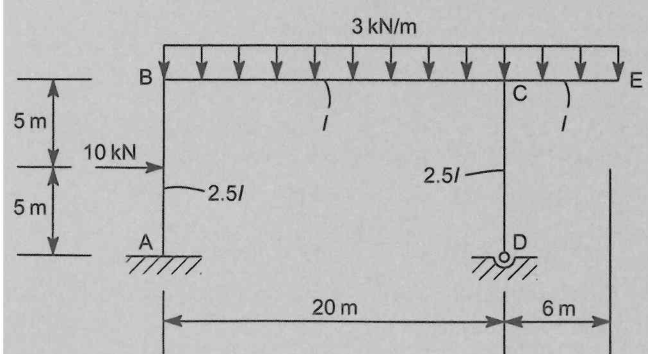


FIGURE 16.36
Portal frame of Ex. 16.16.

$$M^F_{CE} = -\frac{3 \times 6^2}{2} = -54 \text{ kN m}$$

$$M^F_{AB} = -M^F_{BA} = -\frac{10 \times 10}{8} = -12.5 \text{ kN m}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{3 \times 20^2}{12} = -100 \text{ kN m} \quad M^F_{CD} = M^F_{DC} = 0$$

Now, from Eqs (16.28) and (16.30)

$$M_{AB} = -\frac{2 \times 2.5EI}{10}\left(\theta_B - \frac{3}{10}v_1\right) - 12.5 \qquad \text{(i)}$$

$$M_{BA} = -\frac{2 \times 2.5EI}{10}\left(2\theta_B - \frac{3}{10}v_1\right) + 12.5 \qquad \text{(ii)}$$

In Eqs (i) and (ii) we are assuming that the displacement, v_1, is to the right. Furthermore

$$M_{BC} = -\frac{2EI}{20}(2\theta_B + \theta_C) - 100 \qquad \text{(iii)}$$

$$M_{CB} = -\frac{2EI}{20}(2\theta_C + \theta_B) + 100 \qquad \text{(iv)}$$

$$M_{CD} = -\frac{2 \times 2.5EI}{10}\left(2\theta_C + \theta_D + \frac{3}{10}v_1\right) \qquad \text{(v)}$$

$$M_{DC} = -\frac{2 \times 2.5EI}{10}\left(2\theta_D + \theta_C + \frac{3}{10}v_1\right) \qquad \text{(vi)}$$

From the equilibrium of the member end moments at the joints

$$M_{BA} + M_{BC} = 0 \quad M_{CB} + M_{CD} - 54 = 0 \quad M_{DC} = 0$$

Substituting in the equilibrium equations for M_{BA}, M_{BC}, etc., from Eqs (i)–(vi) we obtain

$$1.25EI\theta_B + 0.1EI\theta_C - 0.15EIv_1 + 87.5 = 0 \qquad \text{(vii)}$$

$$1.2EI\theta_C + 0.1EI\theta_B + 0.5EI\theta_C + 0.15EIv_1 - 46 = 0 \qquad \text{(viii)}$$

$$EI\theta_D + 0.5EI\theta_C + 0.15EIv_1 = 0 \qquad \text{(ix)}$$

Since there are four unknown displacements we require a further equation for a solution. This may be obtained by considering the overall horizontal equilibrium of the frame. Thus

$$S_{AB} + S_{DC} - 10 = 0$$

in which, from Eq. (16.29)

$$S_{AB} = \frac{6 \times 2.5EI}{10^2}\theta_B - \frac{12 \times 2.5EI}{10^3}v_1 + 5$$

where the last term on the right-hand side is S_{AB}^{F} (= +5 kN), the contribution of the 10 kN horizontal load to S_{AB}. Also

$$S_{DC} = \frac{6 \times 2.5EI}{10^2}(\theta_D + \theta_C) - \frac{12 \times 2.5EI}{10^3}v_1$$

Hence, substituting for S_{AB} and S_{DC} in the equilibrium equations, we have

$$EI\theta_B + EI\theta_D + EI\theta_C - 0.4EIv_1 - 33.3 = 0 \quad \text{(x)}$$

Solving Eqs (vii)−(x) we obtain

$$EI\theta_B = -101.5 \quad EI\theta_C = +73.2 \quad EI\theta_D = -9.8 \quad EIv_1 = -178.6$$

Substituting these values in Eqs (i)−(vi) yields

$$M_{AB} = 11.5 \text{ kN m} \quad M_{BA} = 87.2 \text{ kN m} \quad M_{BC} = -87.2 \text{ kN m}$$
$$M_{CB} = 95.5 \text{ kN m} \quad M_{CD} = -41.5 \text{ kN m}$$
$$M_{DC} = 0 \quad M_{CE} = -54 \text{ kN m}$$

6.10 Moment distribution

xamples 16.15 and 16.16 show that the greater the complexity of a structure, the greater the ımber of unknowns and therefore the greater the number of simultaneous equations requiring soluɔn; hand methods of analysis then become extremely tedious if not impracticable so that alternatives e desirable. One obvious alternative is to employ computer-based techniques but another, quite powful hand method is an iterative procedure known as the *moment distribution method*. The method was ːrived by Professor Hardy Cross and presented in a paper to the ASCE in 1932.

rinciple

onsider the three-span continuous beam shown in Fig. 16.37(a). The beam carries loads that, as we ıve previously seen, will cause rotations, θ_A, θ_B, θ_C and θ_D at the supports as shown in Fig. 16.37(b). In g. 16.37(b), θ_A and θ_C are positive (corresponding to positive moments) and θ_B and θ_D are negative.

Suppose that the beam is clamped at the supports before the loads are applied, thereby ·eventing these rotations. Each span then becomes a fixed beam with moments at each end, i.e. ΞMs. Using the same notation as in the slope−deflection method these moments are $^{F}_{AB}$, M_{BA}^{F}, M_{BC}^{F}, M_{CB}^{F}, M_{CD}^{F} and M_{DC}^{F}. If we now release the beam at the support B, say, the resultant oment at B, $M_{BA}^{F} + M_{BC}^{F}$, will cause rotation of the beam at B until equilibrium is restored; $^{F}_{BA} + M_{BC}^{F}$ is the *out of balance* moment at B. Note that, at this stage, the rotation of the beam at B *is ıt* θ_B. By allowing the beam to rotate to an equilibrium position at B we are, in effect, applying a balıcing moment at B equal to $-(M_{BA}^{F} + M_{BC}^{F})$. Part of this balancing moment will cause rotation in e span BA and part will cause rotation in the span BC. In other words the balancing moment at B ıs been *distributed* into the spans BA and BC, the relative amounts depending upon the *stiffness*, or e resistance to rotation, of BA and BC. This procedure will affect the FEMs at A and C so that they ill no longer be equal to M_{BA}^{F} and M_{CB}^{F}. We shall see later how they are modified.

We now clamp the beam at B in its new equilibrium position and release the beam at, say, C. This

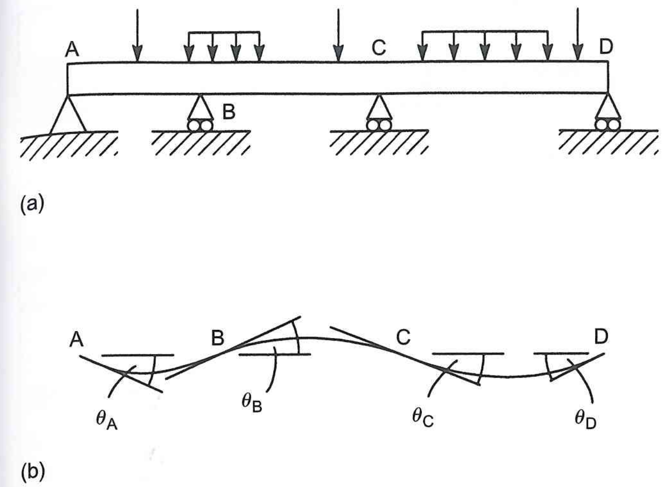


FIGURE 16.37

Principle of the moment distribution method.

position at C. The FEM at D will then be modified and there will now be an out of balance moment at B. The beam is now clamped at C and released in turn at A and D, thereby modifying the moments at B and C.

The beam is now in a position in which it is clamped at each support but in which it has rotated at the supports through angles that are not yet equal to θ_A, θ_B, θ_C and θ_D. Clearly the out of balance moment at each support will not be as great as it was initially since some rotation has taken place; the beam is now therefore closer to the equilibrium state of Fig. 16.37(b). The release/clamping procedure is repeated until the difference between the angle of rotation at each support and the equilibrium state of Fig. 16.37(b) is negligibly small. Fortunately this occurs after relatively few release/clamping operations.

In applying the moment distribution method we shall require the FEMs in the different members of a beam or frame. We shall also need to determine the distribution of the balancing moment at a support into the adjacent spans and also the fraction of the distributed moment which is *carried over* to each adjacent support.

The sign convention we shall adopt for the FEMs is identical to that for the end moments in the slope−deflection method; thus clockwise moments are positive, anticlockwise are negative.

Fixed-end moments

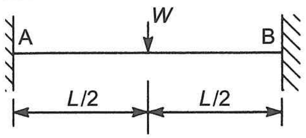
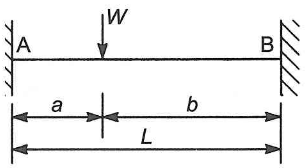
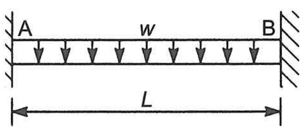
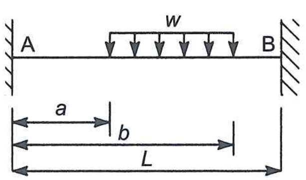
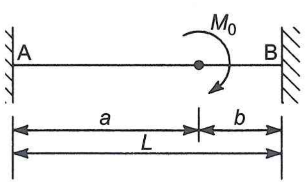
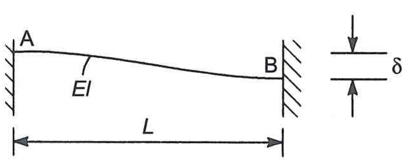
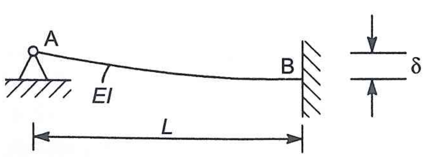
We shall require values of FEMs for a variety of loading cases. It will be useful, therefore, to list them for the more common loading cases; others may be found using the moment-area method described in Section 13.3. Included in Table 16.6 are the results for the fixed beams analysed in Section 13.7.

Stiffness coefficient

A moment applied at a point on a beam causes a rotation of the beam at that point, the angle of rotation being directly proportional to the applied moment (see Eq. (9.19)). Thus for a beam AB and a moment M_{BA} applied at the end B

$$M_{BA} = -K_{AB}\theta_B \quad (16.32)$$

able 16.6

	FEMs (+ / −)		
ocal case	M^F_{AB}	M^F_{BA}	
	$-\frac{WL}{8}$	$+\frac{WL}{8}$	
	$-\frac{Wab^2}{L^2}$	$+\frac{Wa^2b}{L^2}$	
	$-\frac{wL^2}{12}$	$+\frac{wL^2}{12}$	
	$-\frac{w}{L^2}\left[\frac{L^2}{2}(b^2-a^2)-\frac{2}{3}L(b^3-a^3)+\frac{1}{4}(b^4-a^4)\right]$	$+\frac{wb^3}{L^2}\left(\frac{L}{3}-\frac{b}{4}\right)$	
	$+\frac{M_0b}{L^2}(2a-b)$	$+\frac{M_0a}{L^2}(2b-a)$	
	$-\frac{6EI\delta}{L^2}$	$-\frac{6EI\delta}{L^2}$	
	0	$-\frac{3EI\delta}{L^2}$	

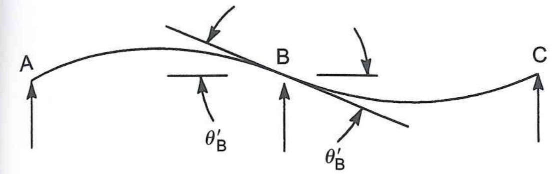


FIGURE 16.38

Determination of DF.

in which K_{AB} ($=K_{BA}$) is the rotational stiffness of the beam AB. The value of K_{AB} depends, as we shall see, upon the support conditions at the ends of the beam. Note that, from Fig. 16.32 a positive M_{BA} decreases θ_B.

Distribution factor

Suppose that in Fig. 16.38 the out of balance moment at the support B in the beam ABC to be distributed into the spans BA and BC is $M_B = (M^F_{BA} + M^F_{BC})$ at the first release. Let M'_{BA} be the fraction of M_B to be distributed into BA and M'_{BC} be the fraction of M_B to be distributed into BC. Suppose also that the angle of rotation at B due to M_B is θ'_B. Then, from Eq. (16.32)

$$M'_{BA} = -K_{BA}\theta'_B \tag{16.33}$$

and

$$M'_{BC} = -K_{BC}\theta'_B \tag{16.34}$$

but

$$M'_{BA} + M'_{BC} + M_{BA} = 0$$

Note that M'_{BA} and M'_{BC} are fractions of the balancing moment while M_B is the out of balance moment. Substituting in this equation for M'_{BA} and M'_{BC} from Eqs (16.33) and (16.34)

$$-\theta'_B(K_{BA} + K_{BC}) = -M_B$$

so that

$$\theta'_B = \frac{M_B}{K_{BA} + K_{BC}} \tag{16.35}$$

Substituting in Eqs (16.33) and (16.34) for θ'_B from Eq. (16.35) we have

$$M'_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}}(-M_B) \quad M'_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}}(-M_B) \tag{16.36}$$

The terms $K_{BA}/(K_{BA} + K_{BC})$ and $K_{BC}/(K_{BA} + K_{BC})$ are the *distribution factors* (DFs) at the support B.

Stiffness coefficients and carry over factors

We shall now derive values of stiffness coefficient (K) and carry over factor (COF) for a number of support and loading conditions. These will be of use in the solution of a variety of problems. For this purpose we use the slope–deflection equations, Eqs (16.28) and (16.30). Thus for a span AB of a beam

$$M_{AB} = -\frac{2EI}{L}\left[2\theta_A + \theta_B + \frac{3}{L}(v_A - v_B)\right]$$

and

$$M_{BA} = -\frac{2EI}{L}\left[2\theta_B + \theta_A + \frac{3}{L}(v_A - v_B)\right]$$

In some problems we shall be interested in the displacement of one end of a beam span relative to the other, i.e. the effect of a sinking support. Thus for, say $v_A = 0$ and $v_B = \delta$ (the final two load cases in Table 16.6) the above equations become

$$M_{AB} = -\frac{2EI}{L}\left(2\theta_A + \theta_B - \frac{3}{L}\delta\right) \tag{16.37}$$

and

$$M_{BA} = -\frac{2EI}{L}\left(2\theta_B + \theta_A - \frac{3}{L}\delta\right) \tag{16.38}$$

Rearranging Eqs (16.37) and (16.38) we have

$$2\theta_A + \theta_B - \frac{3}{L}\delta = -\frac{L}{2EI}M_{AB} \tag{16.39}$$

and

$$2\theta_B + \theta_A - \frac{3}{L}\delta = -\frac{L}{2EI}M_{BA} \tag{16.40}$$

Equations (16.39) and (16.40) may be expressed in terms of various combinations of θ_A, θ_B and δ. Thus subtracting Eq. (16.39) from Eq. (16.40) and rearranging we obtain

$$\theta_B - \theta_A = -\frac{L}{2EI}(M_{BA} - M_{AB}) \tag{16.41}$$

Multiplying Eq. (16.39) by 2 and subtracting from Eq. (16.40) gives

$$\frac{\delta}{L} - \theta_A = -\frac{L}{6EI}(M_{BA} - 2M_{AB}) \tag{16.42}$$

Now eliminating θ_A between Eqs (16.39) and (16.40) we have

$$\theta_B - \frac{\delta}{L} = -\frac{L}{6EI}(2M_{BA} - M_{AB}) \tag{16.43}$$

We shall now use Eqs (16.41)–(16.43) to determine stiffness coefficients and COFs for a variety of support and loading conditions at A and B.

CASE 1: A FIXED, B SIMPLY SUPPORTED, MOMENT M_{BA} APPLIED AT B

This is the situation arising when a beam has been released at a support (B) and we require the stiffness coefficient of the span BA so that we can determine the DF; we also require the fraction of the moment, M_{BA}, which is carried over to the support at A. In this case $\theta_A = \delta = 0$ so that, from Eq. (16.42)

$$M_{AB} = \frac{1}{2}M_{BA}$$

Therefore one-half of the applied moment, M_{BA}, is carried over to A so that the COF = 1/2. Now from Eq. (16.43) we have

$$\theta_B = -\frac{L}{6EI}\left(2M_{BA} - \frac{M_{BA}}{2}\right)$$

so that

$$M_{BA} = -\frac{4EI}{L}\theta_B$$

from which (see Eq. (16.32))

$$K_{BA} = \frac{4EI}{L} \quad (= K_{AB})$$

CASE 2: A SIMPLY SUPPORTED, B SIMPLY SUPPORTED, MOMENT M_{BA} APPLIED AT B

This situation arises when we release the beam at an internal support (B) and the adjacent support (A) is an outside support which is pinned and therefore free to rotate. In this case the moment, M_{BA}, does not affect the moment at A, which is always zero; there is, therefore, no carry over from B to A.

From Eq. (16.43)

$$\theta_B = -\frac{L}{6EI}2M_{BA} \quad (M_{AB} = 0)$$

which gives

$$M_{BA} = -\frac{3EI}{L}\theta_B$$

so that

$$K_{BA} = \frac{3EI}{L}(= K_{AB})$$

CASE 3: A AND B SIMPLY SUPPORTED, EQUAL MOMENTS M_{BA} AND $-M_{AB}$ APPLIED AT B AND A

This case is of use in a symmetrical beam that is symmetrically loaded and would apply to the central span. Thus identical operations will be carried out at each end of the central span so that there will be no carry over of moment from B to A or A to B. Also $\theta_B = -\theta_A$ so that from Eq. (16.41)

$$M_{BA} = -\frac{2EI}{L}\theta_B$$

and

$$K_{BA} = \frac{2EI}{L}(= K_{AB})$$

CASE 4: A AND B SIMPLY SUPPORTED, THE BEAM ANTISYMMETRICALLY LOADED SUCH THAT $M_{BA} = M_{AB}$

This case uses the antisymmetry of the beam and loading in the same way that Case 3 uses symmetry. There is therefore no carry over of moment from B to A or A to B and $\theta_A = \theta_B$. Therefore, from Eq. (16.43)

$$M_{BA} = -\frac{6EI}{L}\theta_B$$

so that

$$K_{BA} = \frac{6EI}{L}(= K_{AB})$$

We are now in a position to apply the moment distribution method to beams and frames. Note that the successive releasing and clamping of supports is, in effect, carried out simultaneously in the analysis.

First we shall consider continuous beams.

Continuous beams

EXAMPLE 16.17

Determine the support reactions in the continuous beam ABCD shown in Fig. 16.39; its flexural rigidity EI is constant throughout.

Initially we calculate the FEMs for each of the three spans using the results presented in Table 16.6. Thus

$$M^F_{AB} = -M^F_{BA} = -\frac{8 \times 3^2}{12} = -6.0 \text{ kNm}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{8 \times 2^2}{12} - \frac{20 \times 2}{8} = -7.67 \text{ kNm}$$

$$M^F_{CD} = -M^F_{DC} = -\frac{8 \times 2^2}{12} = -2.67 \text{ kNm}$$

In this particular example certain features should be noted. Firstly, the support at A is a fixed support so that it will not be released and clamped in turn. In other words, the moment at A will always be balanced (by the fixed support) but will be continually modified as the beam at B is released and clamped. Secondly, the support at D is an outside pinned support so that the final moment at D must be zero. We can therefore reduce the amount of computation by balancing the beam at D initially and then leaving the support at D pinned so that there will be no carry over of moment from C to D in the subsequent moment distribution. However, the stiffness coefficient of CD must be modified to allow for this since the span CD will then correspond to Case 2

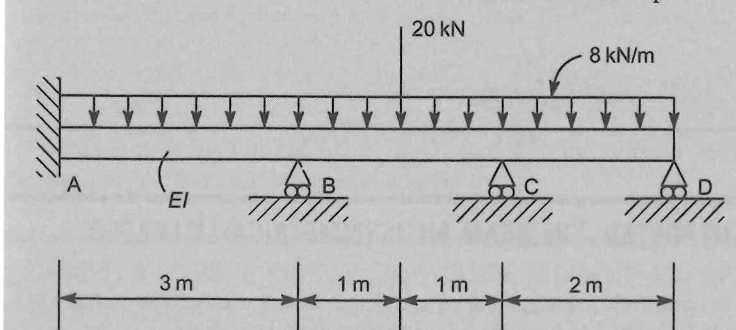


FIGURE 16.39

Beam of Ex. 16.17.

as the beam is released at C and is free to rotate at D. Thus $K_{CD} = K_{DC} = 3EI/L$. All other spans correspond to Case 1 where, as we release the beam at a support, that support is a pinned support while the beam at the adjacent support is fixed. Therefore, for the spans AB and BC, the stiffness coefficients are $4EI/L$ and the COFs are equal to 1/2.

The DFs are obtained from Eq. (16.36). Thus

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{4EI/3}{4EI/3 + 4EI/2} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC}} = \frac{4EI/2}{4EI/3 + 4EI/2} = 0.6$$

$$DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{4EI/3}{4EI/2 + 3EI/2} = 0.57$$

$$DF_{CD} = \frac{K_{CD}}{K_{CB} + K_{CD}} = \frac{3EI/3}{4EI/2 + 3EI/2} = 0.43$$

Note that the sum of the DFs at a support must always be equal to unity since they represent the fraction of the out of balance moment which is distributed into the spans meeting at that support. The solution is now completed as shown in Table 16.7.

Note that there is a rapid convergence in the moment distribution. As a general rule it is sufficient to stop the procedure when the distributed moments are of the order of 2% of the original FEMs. In the table the last moment at C in CD is −0.02 which is 0.75% of the original FEM, while the last moment at B in BC is +0.05 which is 0.65% of the original FEM. We could, therefore, have stopped the procedure at least one step earlier and still have retained sufficient accuracy.

The final reactions at the supports are now calculated from the final support moments and the reactions corresponding to the actual loads, i.e. the free reactions; these are calculated as though each span were simply supported. The procedure is identical to that in Ex. 16.15.

Table 16.7

	A	B		C		D
DFs	–	**0.4**	**0.6**	**0.57**	**0.43**	**1.0**
FEMs	−6.0	+6.0	−7.67	+7.67	−2.67	+2.67
Balance D						−2.67
Carry over					−1.34	
Balance		+0.67	+1.0	−2.09	−1.58	
Carry over	+0.34		−1.05	+0.5		
Balance		+ 0.42	+ 0.63	−0.29	−0.21	
Carry over	+0.21		−0.15	+0.32		
Balance		+ 0.06	+0.09	−0.18	−0.14	
Carry over	+0.03		−0.09	+0.05		
Balance		+0.04	+0.05	−0.03	−0.02	
Final moments	−5.42	+7.19	−7.19	+5.95	−5.95	0

Table 16.8

	A	B		C		D
Free reactions	↑12.0	12.0↑	↑18.0	18.0↑	↑8.0	8.0↑
Final moment reactions	↓0.6	0.6↑	↑0.6	0.6↓	↑2.98	2.98↓
Total reactions (kN)	↑11.4	12.6↑	↑18.6	17.4↑	↑10.98	5.02↑

For example, in Table 16.8 the final moment reactions in AB form a couple to balance the clockwise moment of $7.19 - 5.42 = 1.77$ kN m acting on AB. Thus at A the reaction is $1.77/3.0 = 0.6$ kN acting downwards while at B in AB the reaction is 0.6 kN acting upwards. The remaining final moment reactions are calculated in the same way.

Finally the complete reactions at each of the supports are

$$R_A = 11.4\ \text{kN} \quad R_B = 12.6 + 18.6 = 31.2\ \text{kN}$$

$$R_C = 17.4 + 10.98 = 28.38\ \text{kN} \quad R_D = 5.02\ \text{kN}$$

EXAMPLE 16.18

Calculate the support reactions in the beam shown in Fig. 16.40; the flexural rigidity, *EI*, of the beam is constant throughout.

This example differs slightly from Ex. 16.17 in that there is no fixed support and there is a cantilever overhang at the right-hand end of the beam. We therefore treat the support at A in exactly the same way as the support at D in the previous example. The effect of the cantilever overhang may be treated in a similar manner since we know that the final value of moment at D is $-5 \times 4 = -20$ kN m. We therefore calculate the FEMs M^F_{DE} $(=-20$ kN m) and M^F_{DC}, balance the beam at D, carry over to C and then leave the beam at D balanced and pinned; again the stiffness coefficient, K_{DC}, is modified to allow for this (Case 2).

The FEMs are again calculated using the appropriate results from Table 16.6. Thus

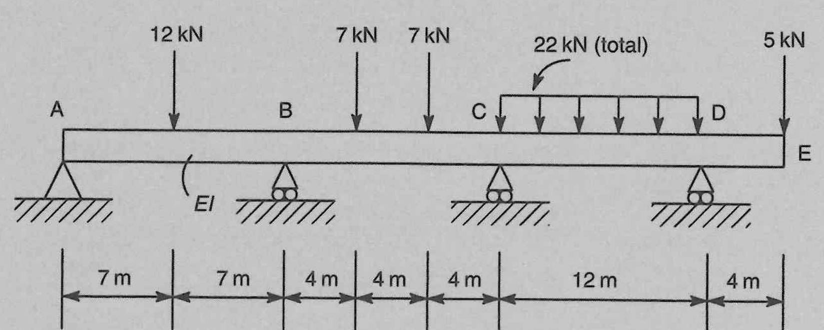


FIGURE 16.40

Beam of Ex. 16.18.

$$M^F_{AB} = -M^F_{BA} = -\frac{12 \times 14}{8} = -21\ \text{kN m}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{7 \times 4 \times 8^2}{12^2} - \frac{7 \times 8 \times 4^2}{12^2} = -18.67\ \text{kN m}$$

$$M^F_{CD} = -M^F_{DC} = -\frac{22 \times 12}{12} = -22\ \text{kN m}$$

$$M^F_{DE} = -5 \times 4 = -20\ \text{kN m}$$

The DFs are calculated as follows

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{3EI/14}{3EI/14 + 4EI/12} = 0.39$$

Hence

$$DF_{BC} = 1 - 0.39 = 0.61$$

$$DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{4EI/12}{4EI/12 + 3EI/12} = 0.57$$

Hence

$$DF_{CD} = 1 - 0.57 = 0.43$$

The solution is completed as follows:

	A	B		C		D		E
DFs	1	0.39	0.61	0.57	0.43	1.0	0	–
FEMs	−21.0	+21.0	−18.67	+18.67	−22.0	+22.0	−20.0	0
Balance A and D	+21.0					−2.0		
Carry over		+10.5			−1.0			
Balance		−5.0	−7.83	+2.47	+1.86			
Carry over			+1.24	−3.92				
Balance		−0.48	−0.76	+2.23	+1.69			
Carry over			+1.12	−0.38				
Balance		−0.44	−0.68	+0.22	+0.16			
Carry over			+0.11	−0.34				
Balance		−0.04	−0.07	+0.19	+0.15			
Final moments	0	+25.54	−25.54	+19.14	−19.14	+20.0	−20.0	0

The support reactions are now calculated in an identical manner to that in Ex. 16.17 and are

$$R_A = 4.18\ \text{kN} \quad R_B = 15.35\ \text{kN} \quad R_C = 17.4\ \text{kN} \quad R_D = 16.0\ \text{kN}$$

EXAMPLE 16.19

Calculate the reactions at the supports in the beam ABCD shown in Fig. 16.41. The flexural rigidity of the beam is constant throughout.

The beam in Fig. 16.41 is symmetrically supported and loaded about its centre line; we may therefore use this symmetry to reduce the amount of computation.

In the centre span, BC, $M^F_{BC} = -M^F_{CB}$ and will remain so during the distribution. This situation corresponds to Case 3, so that if we reduce the stiffness (K_{BC}) of BC to $2EI/L$ there will be no carry over of moment from B to C (or C to B) and we can consider just half the beam. The outside pinned support at A is treated in exactly the same way as the outside pinned supports in Exs 16.17 and 16.18.

The FEMs are

$$M^F_{AB} = -M^F_{BA} = -\frac{5\times 6^2}{12} = -15 \text{ kN m}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{40\times 5}{8} = -25 \text{ kN m}$$

The DFs are

$$DF_{AB} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{3EI/6}{3EI/6+2EI/10} = 0.71$$

Hence

$$DF_{BC} = 1 - 0.71 = 0.29$$

The solution is completed as follows:

	A	B	
DFs	1	0.71	0.29
FEMs	− 15.0	+ 15.0	− 25.0
Balance A	+ 15.0		
Carry over		+ 7.5	
Balance B		+ 1.78	+ 0.72
Final moments	0	+ 24.28	− 24.28

Note that we only need to balance the beam at B once. The use of symmetry therefore leads to a significant reduction in the amount of computation.

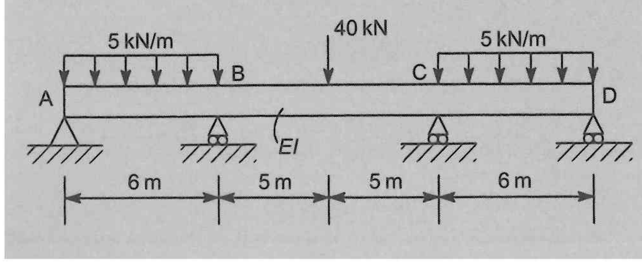


FIGURE 16.41
Symmetrical beam of Ex. 16.19.

The support reactions are now calculated as in Ex. 16.17 and are

$$R_A = R_D = 10.95 \text{ kN}, \quad R_B = R_C = 39.05 \text{ kN}.$$

EXAMPLE 16.20

Calculate the end moments at the supports in the beam shown in Fig. 16.42 if the support at B is subjected to a settlement of 12 mm. Furthermore, the second moment of area of the cross section of the beam is 9×10^6 mm^4 in the span AB and 12×10^6 mm^4 in the span BC; Young's modulus, E, is 200 000 N/mm^2.

In this example the FEMs produced by the applied loads are modified by additional moments produced by the sinking support. Thus, using Table 16.6

$$M^F_{AB} = -\frac{6\times 5^2}{12} - \frac{6\times 200\,000\times 9\times 10^6\times 12}{(5\times 10^3)^2\times 10^6} = -17.7 \text{ kN m}$$

$$M^F_{BA} = +\frac{6\times 5^2}{12} - \frac{6\times 200\,000\times 9\times 10^6\times 12}{(5\times 10^3)^2\times 10^6} = +7.3 \text{ kN m}$$

Since the support at C is an outside pinned support, the effect on the FEMs in BC of the settlement of B is reduced (see the last case in Table 16.6). Thus

$$M^F_{BC} = -\frac{40\times 6}{8} + \frac{3\times 200\,000\times 12\times 10^6\times 12}{(6\times 10^3)^2\times 10^6} = -27.6 \text{ kN m}$$

$$M^F_{CB} = +\frac{40\times 6}{8} = +30.0 \text{ kNm}$$

The DFs are

$$DF_{BA} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{(4E\times 9\times 10^6)/5}{(4E\times 9\times 10^6)/5+(3E\times 12\times 10^6)/6} = 0.55$$

Hence

$$DF_{BC} = 1 - 0.55 = 0.45$$

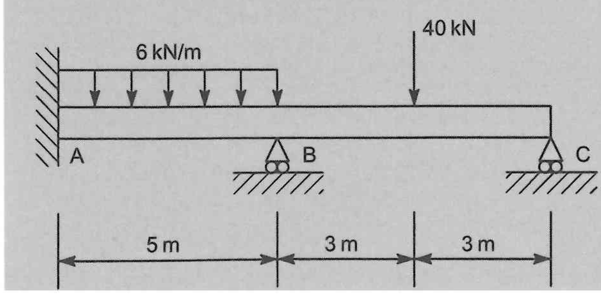


FIGURE 16.42
Beam of Ex. 16.20.

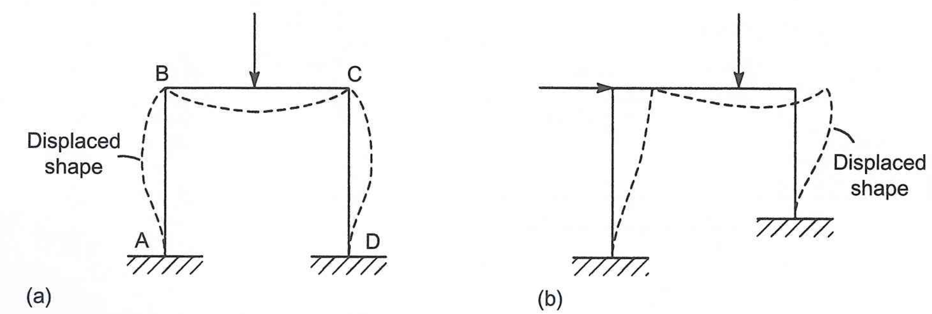
	A	B	B	C
DFs	–	**0.55**	**0.45**	**1.0**
FEMs	−17.7	+7.3	−27.6	+30.0
Balance C				−30.0
Carry over			−15.0	
Balance B		+19.41	+15.89	
Carry over	+9.71			
Final moments	−7.99	+26.71	−26.71	0

Note that in this example balancing the beam at B has a significant effect on the fixing moment t A; we therefore complete the distribution after a carry over to A.

5.11 Portal frames

rtal frames fall into two distinct categories. In the first the frames, such as that shown in Fig. 16.43 , are symmetrical in geometry and symmetrically loaded, while in the second (Fig. 16.43(b)) the mes are unsymmetrical due either to their geometry, the loading or a combination of both. The dis- cements in the symmetrical frame of Fig. 16.43(a) are such that the joints at B and C remain in ir original positions (we are ignoring axial and shear displacements and we assume that the joints nain rigid so that the angle between adjacent members at a joint is unchanged by the loading). In unsymmetrical frame there are additional displacements due to side sway or *sway* as it is called. is sway causes additional moments at the ends of the members which must be allowed for in the lysis.

Initially we shall consider frames in which there is no sway. The analysis is then virtually identical that for continuous beams with only, in some cases, the added complication of more than two mem- s meeting at a joint.



URE 16.43

EXAMPLE 16.21

Obtain the bending moment diagram for the frame shown in Fig. 16.44; the flexural rigidity *EI* is the same for all members.

In this example the frame is unsymmetrical but sway is prevented by the member BC which is fixed at C. Also, the member DA is fixed at D while the member EB is pinned at E.

The FEMs are calculated using the results of Table 16.6 and are

$$M^F_{AD} = M^F_{DA} = 0 \quad M^F_{BE} = M^F_{EB} = 0$$

$$M^F_{AB} = -M^F_{BA} = -\frac{12 \times 4 \times 8^2}{12^2} - \frac{12 \times 4 \times 4^2}{12^2} = -32\ \text{kN m}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{1 \times 16^2}{12} = -21.3\ \text{kN m}$$

Since the vertical member EB is pinned at E, the final moment at E is zero. We may therefore treat E as an outside pinned support, balance E initially and reduce the stiffness coefficient, K_{BE}, as before. However, there is no FEM at E so that the question of balancing E initially does not arise. The DFs are now calculated

$$DF_{AD} = \frac{K_{AD}}{K_{AD} + K_{AB}} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

Hence

$$DF_{AB} = 1 - 0.5 = 0.5$$

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC} + K_{BE}} = \frac{4EI/12}{4EI/12 + 4EI/16 + 3EI/12} = 0.4$$

$$DF_{BC} = \frac{K_{BC}}{K_{BA} + K_{BC} + K_{BE}} = \frac{4EI/16}{4EI/12 + 4EI/16 + 3EI/12} = 0.3$$

Hence

$$DF_{BE} = 1 - 0.4 - 0.3 = 0.3$$

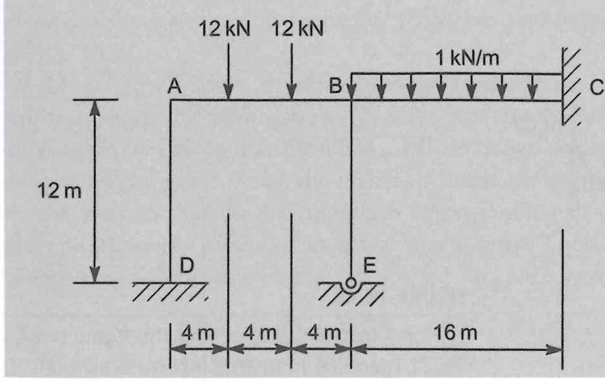


FIGURE 16.44

Beam of Ex. 16.21.

The solution is now completed below.

Joint	D	A		B			C	E
Member	DA	AD	AB	BA	BE	BC	CB	EB
DFs	–	0.5	0.5	0.4	0.3	0.3	–	1.0
FEMs	0	0	−32.0	+32.0	0	−21.3	+21.3	0
Balance A and B		+16.0	+16.0	−4.3	−3.2	−3.2		
Carry over	+8.0		−2.15	+8.0			−1.6	
Balance		+1.08	+1.08	−3.2	−2.4	−2.4		
Carry over	+0.54		−1.6	+0.54			−1.2	
Balance		+0.8	+0.8	−0.22	−0.16	−0.16		
Carry over	+0.4		−0.11	+0.4			−0.08	
Balance		+0.05	+0.06	−0.16	−0.12	−0.12		
Final moments	+8.94	+ 17.93	−17.93	+33.08	−5.88	−27.18	+18.42	0

The bending moment diagram is shown in Fig. 16.45 and is drawn on the tension side of each member. The bending moment distributions in the members AB and BC are determined by superimposing the fixing moment diagram on the free bending moment diagram, i.e. the bending moment diagram obtained by supposing that AB and BC are simply supported.

We shall now consider frames that are subject to sway. For example, the frame shown in Fig. 16.46 (a), although symmetrical itself, is unsymmetrically loaded and will therefore sway. Let us suppose that the final end moments in the members of the frame are M_{AB}, M_{BA}, M_{BC}, etc. Since we are assuming a linearly elastic system we may calculate the end moments produced by the applied loads assuming that the frame does not sway, then calculate the end moments due solely to sway and superimpose the two cases. Thus

$$M_{AB} = M_{AB}^{NS} + M_{AB}^{S} \quad M_{BA} = M_{BA}^{NS} + M_{BA}^{S}$$

and so on, in which M_{AB}^{NS} is the end moment at A in the member AB due to the applied loads, assuming that sway is prevented, while M_{AB}^{S} is the end moment at A in the member AB produced by sway only, and so on for M_{BA}, M_{BC}, etc.

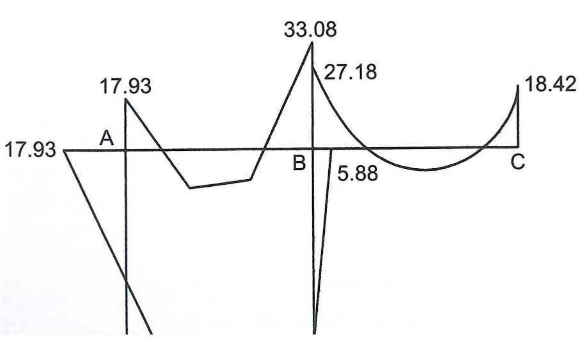


FIGURE 16.45

Bending moment diagram for the frame of Ex. 16.21 (bending moments (kN m) drawn on

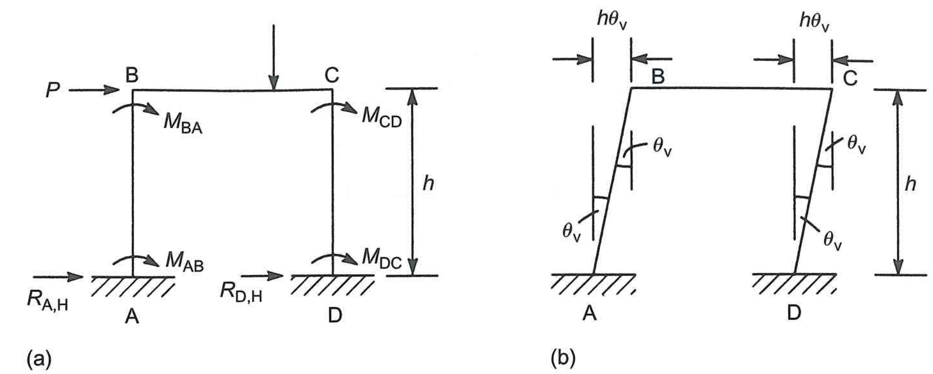


FIGURE 16.46

Calculation of sway effect in a portal frame.

We shall now use the principle of virtual work (Section 15.2) to establish a relationship between the final end moments in the member and the applied loads. Thus we impose a small virtual displacement on the frame comprising a rotation, θ_v, of the members AB and DC as shown in Fig. 16.46(b). This displacement *should not be confused with the sway of the frame* which may, or may not, have the same form depending on the loads that are applied. In Fig. 16.46(b) the members are rotating as rigid links so that the internal moments in the members do no work. Therefore the total virtual work comprises external virtual work only (the end moments M_{AB}, M_{BA}, etc. are externally applied moments as far as each frame member is concerned) so that, from the principle of virtual work

$$M_{AB}\theta_v + M_{BA}\theta_v + M_{CD}\theta_v + M_{DC}\theta_v + Ph\theta_V = 0$$

Hence

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} + Ph = 0 \tag{16.44}$$

Note that, in this case, the member BC does not rotate so that the end moments M_{BC} and M_{CB} do no virtual work. Now substituting for M_{AB}, M_{BA}, etc. in Eq. (16.44) we have

$$M_{AB}^{NS} + M_{AB}^{S} + M_{BA}^{NS} + M_{BA}^{S} + M_{CD}^{NS} + M_{CD}^{S} + M_{DC}^{NS} + M_{DC}^{S} + Ph = 0 \tag{16.45}$$

in which the no-sway end moments, M_{AB}^{NS}, etc., are found in an identical manner to those in the frame of Ex. 16.21.

Let us now impose an arbitrary sway on the frame; this can be of any convenient magnitude. The arbitrary sway and moments, M_{AB}^{AS}, M_{BA}^{AS}, etc., are calculated using the moment distribution method in the usual way except that the FEMs will be caused solely by the displacement of one end of a member relative to the other. Since the system is linear the member end moments will be directly proportional to the sway so that the end moments corresponding to the actual sway will be directly proportional to the end moments produced by the arbitrary sway. Thus, $M_{AB}^{S} = kM_{AB}^{AS}, M_{BA}^{S} = kM_{BA}^{AS}$, etc. in which k is a constant. Substituting in Eq. (16.45) for M_{AB}^{S}, M_{BA}^{S}, etc. we obtain

$$M_{AB}^{NS} + M_{BA}^{NS} + M_{CD}^{NS} + M_{CD}^{NS} + k(M_{AB}^{AS} + M_{BA}^{AS} + M_{CD}^{AS} + M_{DC}^{AS}) + Ph = 0 \tag{16.46}$$

Substituting the calculated values of M_{AB}^{AS}, M_{AB}^{AS}, etc. in Eq. (16.46) gives k. The actual sway ıoments M_{AB}^{S}, etc., follow as do the final end moments, $M_{AB}(=M_{AB}^{NS}+M_{AB}^{S}$, etc).

An alternative method of establishing Eq. (16.44) is to consider the equilibrium of the members AB nd DC. Thus, from Fig. 16.46(a) in which we consider the moment equilibrium of the member AB ɔout B we have

$$R_{A,H}h - M_{AB} - M_{BA} = 0$$

hich gives

$$R_{A,H} = \frac{M_{AB} + M_{BA}}{h}$$

Similarly, by considering the moment equilibrium of DC about C

$$R_{D,H} = \frac{M_{DC} + M_{CD}}{h}$$

Now, from the horizontal equilibrium of the frame

$$R_{A,H} + R_{D,H} + P = 0$$

ɔ that, substituting for $R_{A,H}$ and $R_{D,H}$ we obtain

$$M_{AB} + M_{BA} + M_{DC} + M_{CD} + Ph = 0$$

hich is Eq. (16.44).

EXAMPLE 16.22

Obtain the bending moment diagram for the portal frame shown in Fig. 16.47(a). The flexural rigidity of the horizontal member BC is $2EI$ while that of the vertical members AB and CD is EI.

First we shall determine the end moments in the members assuming that the frame does not sway. The corresponding FEMs are found using the results in Table 16.6 and are as follows:

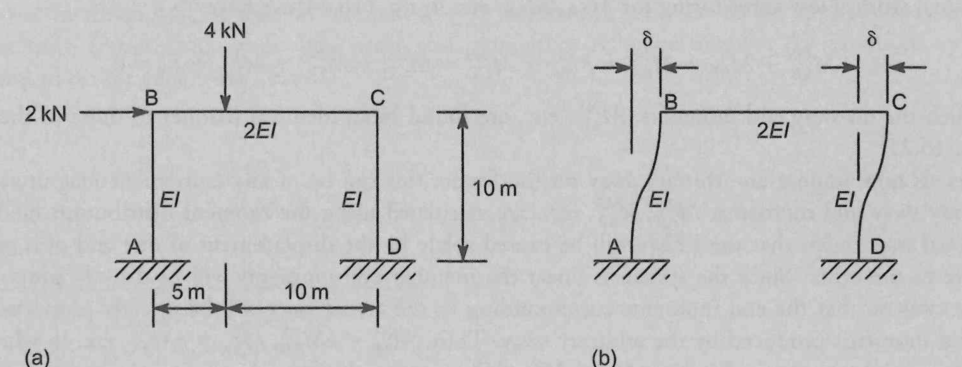


FIGURE 16.47

Portal frame of Ex. 16.22.

$$M_{AB}^F = M_{BA}^F = 0 \quad M_{CD}^F = M_{DC}^F = 0$$

$$M_{BC}^F = -\frac{4\times5\times10^2}{15^2} = -8.89 \text{ kN m}$$

$$M_{CB}^F = +\frac{4\times10\times5^2}{15^2} = +4.44 \text{ kN m}$$

The DFs are

$$DF_{BA} = \frac{K_{BA}}{K_{BA}+K_{BC}} = \frac{4EI/10}{4EI/10 + 4\times2EI/15} = 0.43$$

Hence

$$DF_{BC} = 1 - 0.43 = 0.57$$

From the symmetry of the frame, $DF_{CB} = 0.57$ and $DF_{CD} = 0.43$.

The no-sway moments are determined in the table overleaf. We now assume that the frame sways by an arbitrary amount, δ, as shown in Fig. 16.47(b). Since we are ignoring the effect of axial strains, the horizontal movements of B and C are both δ. The FEMs corresponding to this sway are then (see Table 16.6)

$$M_{AB}^F = M_{BA}^F = -\frac{6EI\delta}{10^2} = M_{DC}^F = M_{CD}^F$$

$$M_{BC}^F = M_{CB}^F = 0$$

Suppose that $\delta = 100\times10^2/6EI$. Then

$$M_{AB}^F = M_{BA}^F = M_{DC}^F = M_{CD}^F = -100 \text{ kN m} \quad \text{(a convenient)}$$

The DFs for the members are the same as those in the no-sway case since they are functions of the member stiffness. We now obtain the member end moments corresponding to the arbitrary sway.

No-sway case

	A	B		C		D
DFs	–	0.43	0.57	0.57	0.43	–
FEMs	0	0	−8.89	+4.44	0	0
Balance		+3.82	+5.07	−2.53	−1.91	
Carry over	+1.91		−1.26	+2.53		−0.95
Balance		+0.54	+0.72	−1.44	−1.09	
Carry over	+0.27		−0.72	+0.36		−0.55
Balance		+0.31	+0.41	−0.21	−0.15	
Carry over	+0.15		−0.11	+0.21		−0.08
Balance		+0.05	+0.06	−0.12	−0.09	
Carry over	+0.03		−0.06	+0.03		−0.05
Balance		+0.03	+0.03	−0.02	−0.01	
Final moments (M^{NS})	+2.36	+4.75	−4.75	+3.25	−3.25	−1.63

Sway case

	A	B		C		D
DFs	–	**0.43**	**0.57**	**0.57**	**0.43**	–
FEMs	−100	−100	0	0	−100	−100
Balance		+43	+57	+57	+43	
Carry over	+21.5		+28.5	+28.5		+21.5
Balance		−12.3	−16.2	−16.2	−12.3	
Carry over	−6.2		−8.1	−8.1		−6.2
Balance		+3.5	+4.6	+4.6	+3.5	
Carry over	+1.8		+2.3	+2.3		+1.8
Balance		−1.0	−1.3	−1.3	−1.0	
Final arbitrary sway moments (M^{AS})	−82.9	−66.8	+66.8	+66.8	−66.8	−82.9

Comparing the frames shown in Figs 16.47 and 16.46 we see that they are virtually identical. We may therefore use Eq. (16.46) directly. Thus, substituting for the no-sway and arbitrary-sway end moments we have

$$2.36 + 4.75 - 3.25 - 1.63 + k(-82.9 - 66.8 - 82.9) + 2 \times 10 = 0$$

which gives

$$k = 0.074$$

The actual sway moments are then

$$M^S_{AB} = kM^{AS}_{AB} = 0.074 \times (-82.9) = -6.14 \text{ kN m}$$

Similarly

$$M^S_{BA} = -4.94 \text{ kN m} \quad M^S_{BC} = 4.94 \text{ kN m} \quad M^S_{CB} = 4.94 \text{ kN m}$$
$$M^S_{CD} = -4.94 \text{ kN m} \quad M^S_{DC} = -6.14 \text{ kN m}$$

Thus the final end moments are

$$M_{AB} = M^{NS}_{AB} + M^S_{AB} = 2.36 - 6.14 = -3.78 \text{ kNm}$$

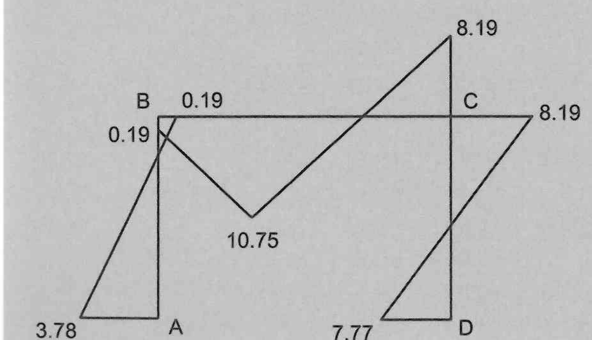


FIGURE 16.48
Bending moment diagram for the portal frame of Ex. 16.22. Bending moments (kNm) drawn on tension side of members.

Similarly

$$M_{BA} = -0.19 \text{ kN m} \quad M_{BC} = -0.19 \text{ kN m} \quad M_{CB} = 8.19 \text{ kN m}$$
$$M_{CD} = -8.19 \text{ kN m} \quad M_{DC} = -7.77 \text{ kN m}$$

The bending moment diagram is shown in Fig. 16.48 and is drawn on the tension side of the members.

EXAMPLE 16.23

Calculate the end moments in the members of the frame shown in Fig. 16.49. All members have the same flexural rigidity, *EI*; note that the member CD is pinned to the foundation at D.

Initially, the FEMs produced by the applied loads are calculated. Thus, from Table 16.6

$$M^F_{BA} = -M^F_{BA} = -\frac{40 \times 6}{8} = -30 \text{ kN m}$$

$$M^F_{BC} = -M^F_{CB} = -\frac{20 \times 6^2}{12} = -60 \text{ kN m}$$

$$M^F_{CD} = M^F_{DC} = 0$$

The DFs are calculated as before. Note that the length of the member $CD = \sqrt{6^2 + 4.5^2} = 7.5$ m.

$$DF_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{4EI/6}{4EI/6 + 4EI/6} = 0.5$$

Hence

$$DF_{BC} = 1 = -0.5 = 0.5$$

$$DF_{CB} = \frac{K_{CB}}{K_{CB} + K_{CD}} = \frac{4EI/6}{4EI/6 + 3EI/7.5} = 0.625$$

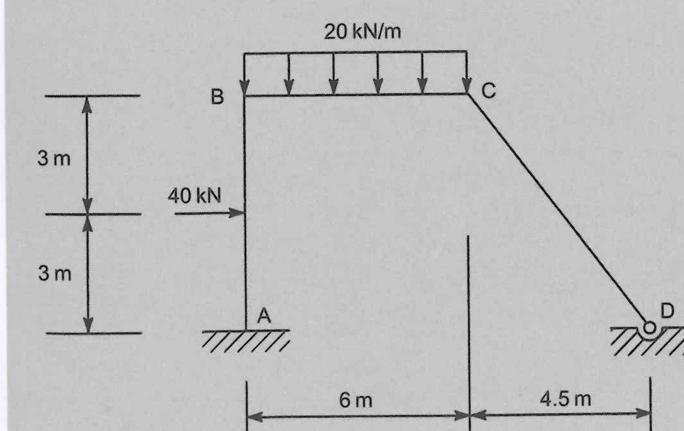


FIGURE 16.49
Frame of Ex. 16.23.

Therefore

$$\mathrm{DF_{CD}} = 1 - 0.625 = 0.375$$

No-sway case

	A	B		C		D
DFs	–	0.5	0.5	0.625	0.375	1.0
FEMs	−30.0	+30.0	−60.0	+60.0	0	0
Balance		+15.0	+15.0	−37.5	−22.5	
Carry over	+7.5		−18.8	+7.5		
Balance		+9.4	+9.4	−4.7	−2.8	
Carry over	+4.7		−2.4	+4.7		
Balance		+1.2	+1.2	−2.9	−1.8	
Carry over	+0.6		−1.5	+0.6		
Balance		+0.75	+0.75	−0.38	−0.22	
Final moments(M^{NS})	−17.2	+56.35	−56.35	+27.32	−27.32	0

Unlike the frame in Ex. 16.22 the frame itself in this case is unsymmetrical. Therefore the geometry of the frame, after an imposed arbitrary sway, will not have the simple form shown in Fig. 16.47 (b). Furthermore, since the member CD is inclined, an arbitrary sway will cause a displacement of the joint C relative to the joint B. This also means that in the application of the principle of virtual work a virtual rotation of the member AB will result in a rotation of the member BC, so that the end moments M_{BC} and M_{CB} will do work; Eq. (16.46) cannot, therefore, be used in its existing form. In this situation we can make use of the geometry of the frame after an arbitrary virtual displacement to deduce the relative displacements of the joints produced by an imposed arbitrary sway; the FEMs due to the arbitrary sway may then be calculated.

Figure 16.50 shows the displaced shape of the frame after a rotation, θ, of the member AB. This diagram will serve, as stated above, to deduce the FEMs due to sway and also to establish a virtual

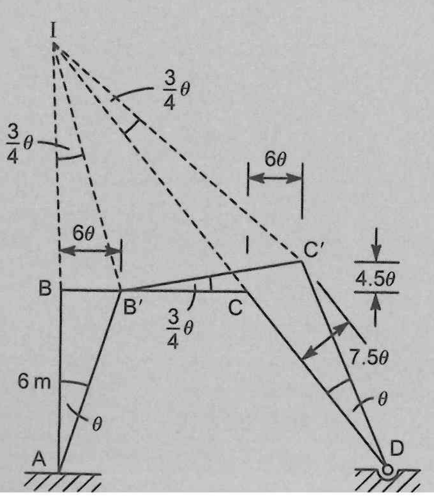


FIGURE 16.50

Arbitrary sway and virtual displacement geometry of frame of Ex. 16.23

work equation similar to Eq. (16.46). It is helpful, when calculating the rotations of the different members, to employ an instantaneous centre, I. This is the point about which the triangle IBC rotates as a rigid body to IB′C′; thus all sides of the triangle rotate through the same angle which, since BI = 8 m (obtained from similar triangles AID and BIC), is $3\theta/4$. The relative displacements of the joints are then as shown.

The FEMs due to the arbitrary sway are, from Table 16.6 and Fig. 16.50

$$M^{F}_{AB} = M^{F}_{BA} = -\frac{6EI(6\theta)}{6^2} = -EI\theta$$

$$M^{F}_{BC} = M^{F}_{CB} = -\frac{6EI(4.5\theta)}{6^2} = +0.75EI\theta$$

$$M^{F}_{CD} = -\frac{3EI(7.5\theta)}{7.5^2} = -0.4EI\theta$$

If we impose an arbitrary sway such that $EI\theta = 100$ we have

$$M^{F}_{AB} = M^{F}_{BA} = -100 \text{ kN m} \quad M^{F}_{BC} = M^{F}_{CB} = +75 \text{ kN m} \quad M^{F}_{CD} = -40 \text{ kN m}$$

Now using the principle of virtual work and referring to Fig. 16.50 we have

$$M_{AB}\theta + M_{BA}\theta + M_{BC}\theta\left(\frac{-3\theta}{4}\right) + M_{CB}\left(\frac{-3\theta}{4}\right)$$
$$+ M_{CD}\theta + 40\left(\frac{6\theta}{2}\right) + 20 \times 6\left(\frac{-4.5\theta}{2}\right) = 0$$

Sway case

	A	B		C		D
DFs	–	0.5	0.5	0.625	0.375	1.0
FEMs	−100	−100	+75	+75	−40	0
Balance		+12.5	+12.5	−21.9	−13.1	
Carry over	+6.3		−10.9	+6.3		
Balance		+5.45	+5.45	−3.9	−2.4	
Carry over	+2.72		−1.95	+2.72		
Balance		−0.97	+0.97	−1.7	−1.02	
Carry over	+0.49		−0.85	+0.49		
Balance		+0.43	+0.43	−0.31	−0.18	
Final arbitrary sway moments(M^{AS})	−90.49	−80.65	+80.65	+56.7	−56.7	

Hence

$$4(M_{AB} + M_{BA} + M_{CD}) - 3(M_{BC} + M_{CB}) - 600 = 0 \qquad \text{(i)}$$

Now replacing M_{AB}, etc., by $M^{NS}_{AB} + kM^{AS}_{AB}$, etc., Eq (i) becomes

$$4(M_{AB}^{NS} + M_{BA}^{NS} + M_{CD}^{NS}) - 3(M_{BC}^{NS} + M_{CB}^{NS}) + k[4(M_{AB}^{AS} + M_{BA}^{AS} + M_{CD}^{AS})$$
$$- 3(M_{BC}^{AS} + M_{CB}^{AS})] - 600 = 0$$

Substituting the values of M_{AB}^{NS} and M_{AB}^{AS}, etc., we have

$$4(-17.2 + 56.35 - 27.32) - 3(-56.35 + 27.32)$$
$$+ k[4(-90.49 - 80.65 - 56.7) - 3(80.65 + 56.7)] - 600 = 0$$

from which $k = -0.352$. The final end moments are calculated from $M_{AB} = M_{AB}^{NS} - 0.352M_{AB}^{AS}$, etc., and are given below.

	AB	BA	BC	CB	CD	DC
No-sway moments	−17.2	+56.4	−56.4	+27.3	−27.3	0
Sway moments	+31.9	+28.4	−28.4	−20.0	+20.0	0
Final moments	+14.7	+84.8	−84.8	+7.3	−7.3	0

The methods described in this chapter are hand methods of analysis although they are fundamental, particularly the slope–deflection method, to the computer-based matrix methods of analysis which are described in Chapter 17.

PROBLEMS

P.16.1 Determine the degrees of static and kinematic indeterminacy in the plane structures shown in Fig. P.16.1.

Ans. (a) $n_s = 3$, $n_k = 6$, (b) $n_s = 1$, $n_k = 2$, (c) $n_s = 2$, $n_k = 4$, (d) $n_s = 6$, $n_k = 15$, (e) $n_s = 2$, $n_k = 7$.

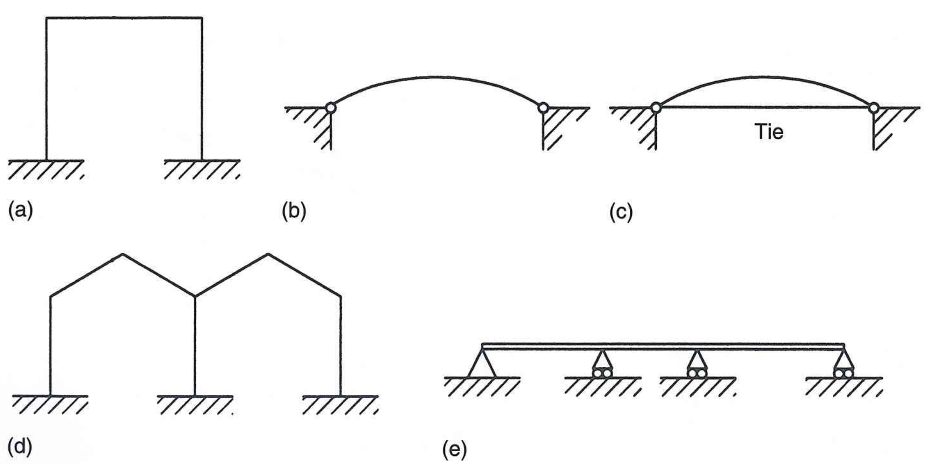


FIGURE P.16.1

P.16.2 Determine the degrees of static and kinematic indeterminacy in the space frames shown in Fig. P.16.2.

Ans. (a) $n_s = 6$, $n_k = 24$, (b) $n_s = 42$, $n_k = 36$, (c) $n_s = 18$, $n_k = 6$.

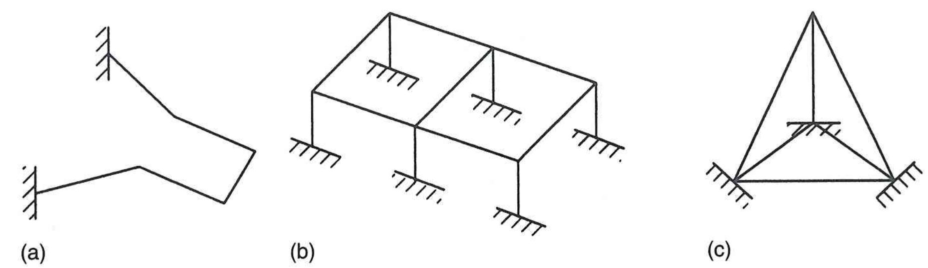


FIGURE P.16.2

P.16.3 Calculate the support reactions in the beam shown in Fig. P.16.3 using a flexibility method.

Ans. $R_A = 3.3$ kN $R_B = 14.7$ kN $R_C = 4.0$ kN $M_A = 2.2$ kNm (hogging).

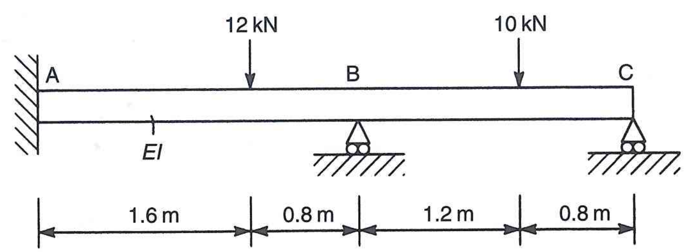


FIGURE P.16.3

P.16.4 Determine the support reactions in the beam shown in Fig. P.16.4 using a flexibility method.

Ans. $R_A = 3.5$ kN $R_B = 9.0$ kN $R_C = 3.5$ kN $M_A = -7$ kN m (hogging) $M_C = -19$ kN m (hogging).

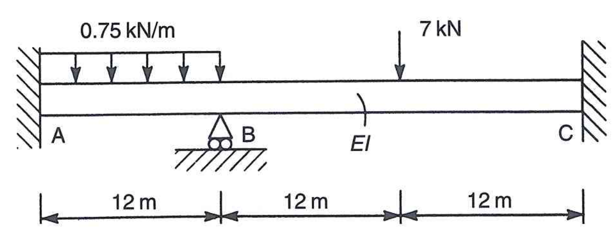


FIGURE P.16.4

P.16.5 Use a flexibility method to determine the support reactions in the beam shown in Fig. P.16.5. The flexural rigidity EI of the beam is constant throughout.

Ans. $R_A = 4.3$ kN $R_B = 15.0$ kN $R_C = 17.8$ kN $R_D = 15.9$ kN.

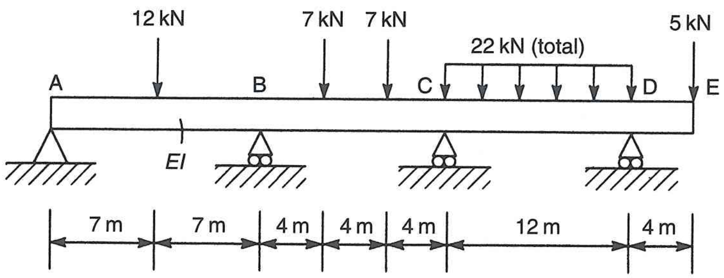


FIGURE P.16.5

P.16.6 Calculate the forces in the members of the truss shown in Fig. P.16.6. The members AC and BD are 30 mm^2 in cross section, all the other members are 20 mm^2 in cross section. The members AD, BC and DC are each 800 mm long; $E = 200\ 000$ N/mm^2.

Ans. AC = 48.2 N BC = 87.6 N BD = −1.8 N CD = 2.1 N AD = 1.1 N.

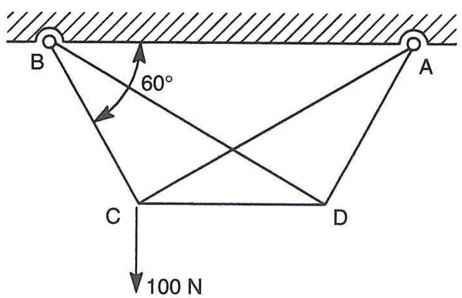


FIGURE P.16.6

P.16.7 Calculate the forces in the members of the truss shown in Fig. P.16.7. The cross-sectional area of all horizontal members is 200 mm^2, that of the vertical members is 100 mm^2 while that of the diagonals is 300 mm^2; E is constant throughout.

Ans. AB = FD = −29.2 kN BC = CD = −29.2 kN AG = GF = 20.8 kN
BG = DG = 41.3 kN AC = FC = −29.4 kN CG = 41.6 kN.

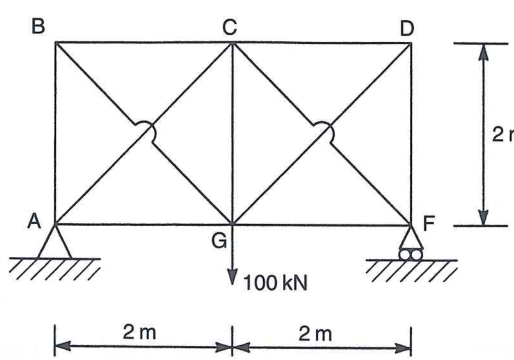


FIGURE P.16.7

P.16.8 Calculate the forces in the members of the truss shown in Fig. P.16.8 and the vertical and horizontal components of the reactions at the supports; all members of the truss have the same cross-sectional properties.

Ans. $R_{A,V} = 67.52$ kN $R_{A,H} = 70.06$ kN $= R_{F,H}$ $R_{F,V} = 32.48$ kN
AB = −32.48 kN AD = −78.31 kN BC = −64.98 kN BD = 72.65 kN
CD = −100.0 kN CE = −64.98 kN DE = 72.65 kN DF = −70.06 kN
EF = −32.49 kN.

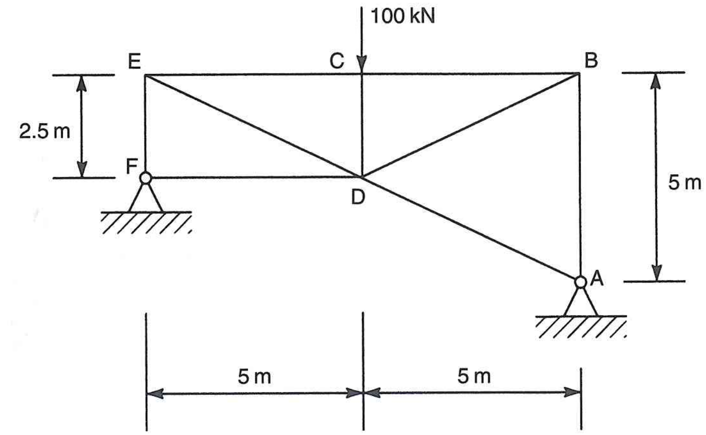
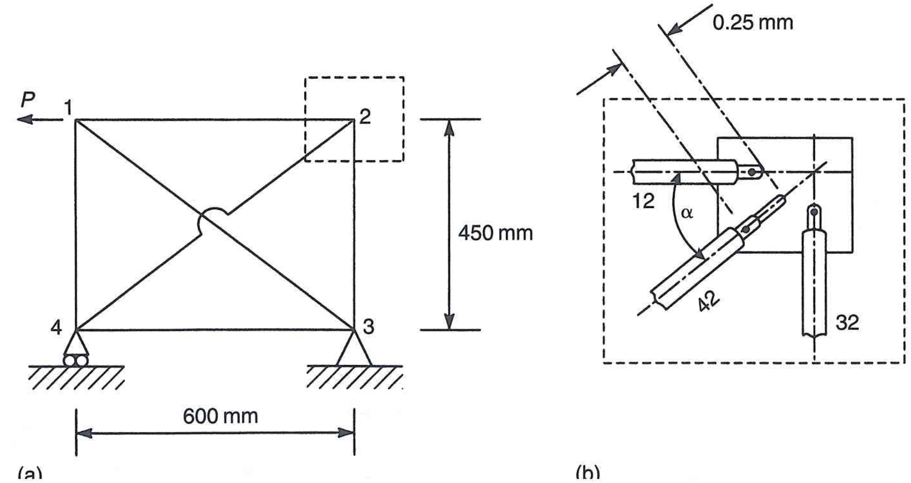


FIGURE P.16.8

P.16.9 The plane truss shown in Fig. P.16.9(a) has one member (24) which is loosely attached at joint 2 so that relative movement between the end of the member and the joint may occur when the framework is loaded. This movement is a maximum of 0.25 mm and takes place only in the direction 24. Figure P.16.9(b) shows joint 2 in detail when the framework is unloaded. Find the value of P at which the member 24 just becomes an effective part of the truss and also the loads in all the members when $P = 10$ kN. All members have a cross-sectional area of 300 mm^2 and a Young's modulus of 70 000 N/mm^2.

Ans. $P = 2.95$ kN 12 = 2.48 kN 23 = 1.86 kN 34 = 2.48 kN 41 = −5.64 kN 13 = 9.4 kN
42 = −3.1 kN.



P.16.10 Figure P.16.10 shows a plane truss pinned to a rigid foundation. All members have the same Young's modulus of 70 000 N/mm^2 and the same cross-sectional area, A, except the member 12 whose cross-sectional area is $1.414A$.

Under some systems of loading, member 14 carries a tensile stress of 0.7 N/mm^2. Calculate the change in temperature which, if applied to member 14 only, would reduce the stress in that member to zero. The coefficient of linear expansion $\alpha = 24 \times 10^{-6}/°$ C.

Ans. 5.5°.

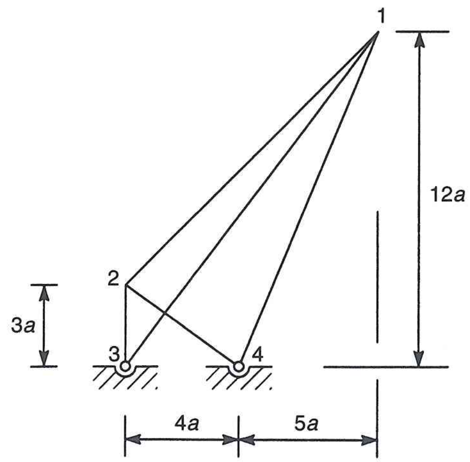


FIGURE P.16.10

P.16.11 The truss shown in Fig. P.16.11 is pinned to a foundation at the points A and B and is supported on rollers at G; all members of the truss have the same axial rigidity $EA = 2 \times 10^9$ N.

Calculate the forces in all the members of the truss produced by a settlement of 15 mm at the support at G.

Ans. GF = 1073.9 kN GH = −536.9 kN FH = −1073.9 kN
FD = 1073.9 kN JH = −1610.8 kN HD = 1073.9 kN
DC = 2147.7 kN CJ = 1073.9 kN JA = −2684.6 kN
AC = −1073.9 kN JD = −1073.9 kN BC = 3221.6 kN.

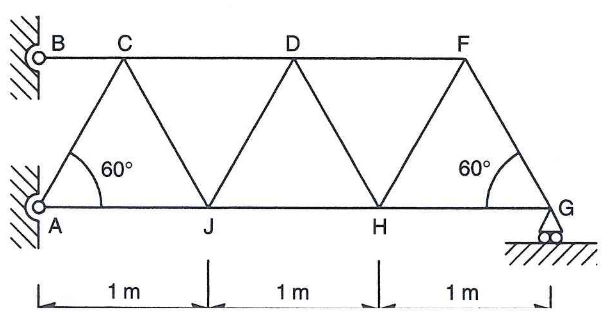


FIGURE P.16.11

P.16.12 Determine the degree of statical indeterminacy of the plane pin-jointed frame shown in Fig. P.16.12 and calculate the forces in the members produced by the vertical load, W. All members have the same length L and axial rigidity AE. Determine also the lack of fit required in the member 45 to reduce the force in the member 41 to zero.

Ans. 14 = −0.81 W, 24 = −0.92W, 25 = +0.23W, 53 = +0.35W, 45 = −0.23W,
46 = −0.58W, 56 = +0.58W, 67 = +0.58W, 47 = −1.15W.
1.73 WL/AE.

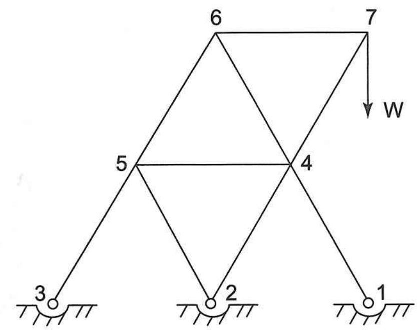


FIGURE P.16.12

P.16.13 Determine the degree of statical indeterminacy of the portal frame shown in Fig. P.16.13 and use the unit load method to determine the reactions at the pinned support at D. All members have the same flexural rigidity EI. Calculate also the bending moments at A, B and C.

Ans. 11.8 kN (vertical, upwards), 3.3 kN (horizontal to the left).
13.2 kNm at A (anticlockwise), 6.98 kNm at B (clockwise), 9.9 kNm at C (anticlockwise).

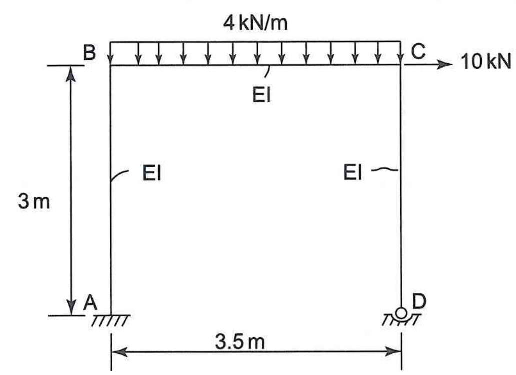


FIGURE P.16.13

P.16.14 The cross-sectional area of the braced beam shown in Fig. P.16.14 is $4A$ and its second moment of area for bending is $Aa^2/16$. All other members have the same cross-sectional area, A, and Young's modulus is E for all members. Find, in terms of w, A, a and E, the vertical displacement of the point D under the loading shown.

Ans. 30 232 $wa^2/3AE$.

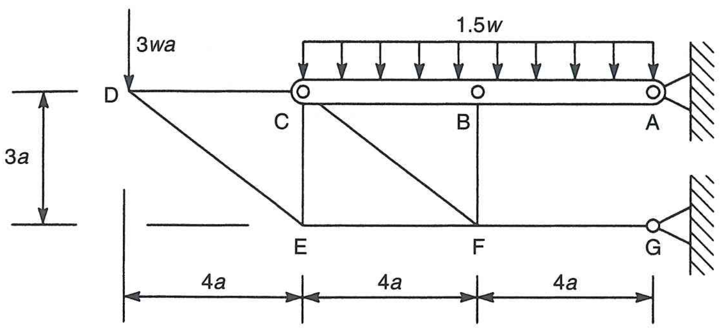


FIGURE P.16.14

P.16.15 Determine the force in the vertical member BD (the king post) in the trussed beam ABC shown in Fig. P.16.15. The cross-sectional area of the king post is 2000 mm^2, that of the beam is 5000 mm^2 while that of the members AD and DC of the truss is 200 mm^2; the second moment of area of the beam is 4.2×10^6 mm^4 and Young's modulus, E, is the same for all members.

Ans. 91.6 kN.

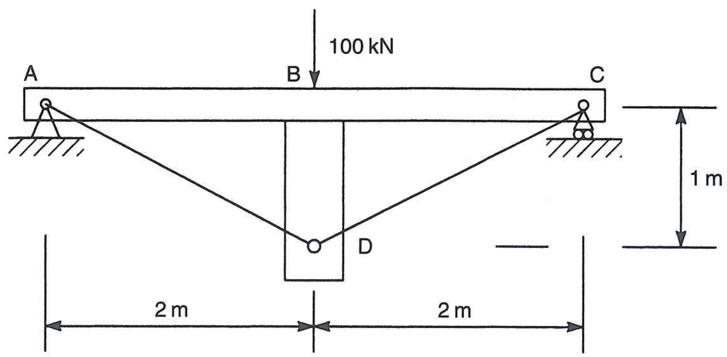


FIGURE P.16.15

P.16.16 Determine the distribution of bending moment in the frame shown in Fig. P.16.16.

Ans. $M_B = 7\,wL^2/45$ $M_C = 8\,wL^2/45$. Parabolic distribution on AB, linear on BC and CD.

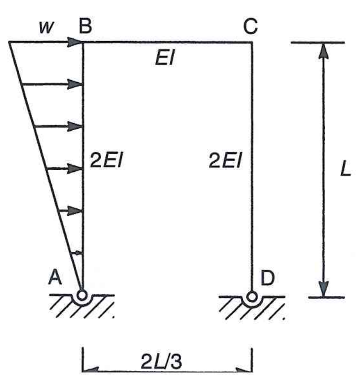


FIGURE P.16.16

P.16.17 Use the flexibility method to determine the end moments in the members of the portal frame shown in Fig. P.16.17. The flexural rigidity of the horizontal member BC is $2EI$ while that of the vertical members AB and CD is EI.

Ans. $M_{AB} = -3.63$ kNm $M_{BA} = -M_{BC} = -0.07$ kNm $M_{CB} = -M_{CD} = 8.28$ kNm $M_{DC} = -8.02$ kN m M (at vert. load) = 10.62 kN m (sagging).

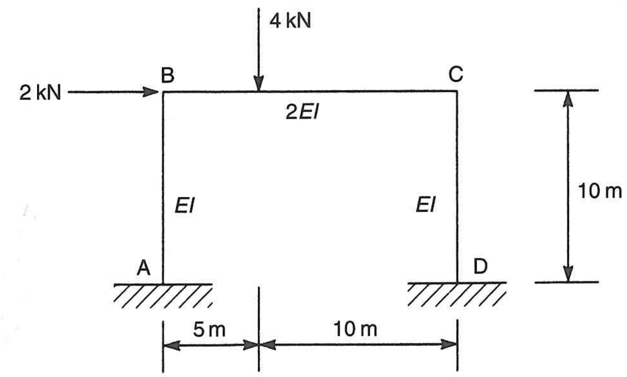


FIGURE P.16.17

P.16.18 Calculate the end moments in the members of the frame shown in Fig. P.16.18 using the flexibility method; all members have the same flexural rigidity, EI.

Ans. $M_{AB} = 14.8$ kNm $M_{BA} = -M_{BC} = 84.8$ kNm $M_{CB} = -M_{CD} = 7.0$ kNm $M_{DC} = 0$.

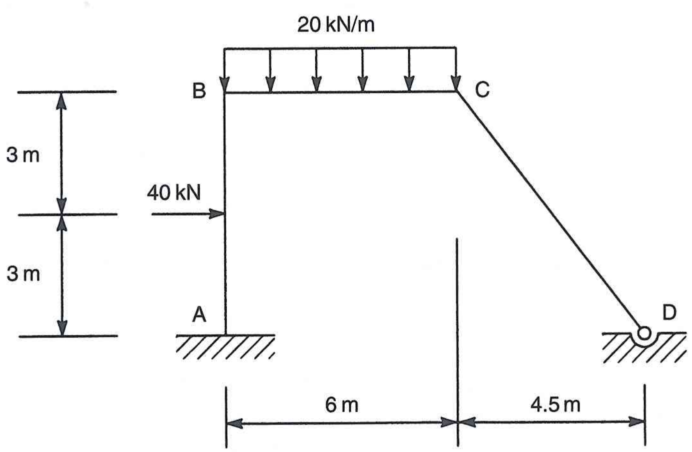


FIGURE P.16.18

P.16.19 The two-pinned circular arch shown in Fig. P.16.19 carries a uniformly distributed load of 15 kN/m over the half-span AC. Calculate the support reactions and the bending moment at the crown C.

Ans. R = 34.1 kN R = 11.4 kN R = R = 17.7 kN $M_C = 3.6$ kNm

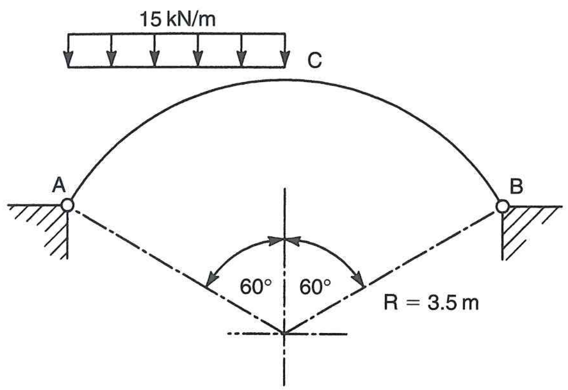


FIGURE P.16.19

16.20 The two-pinned parabolic arch shown in Fig. P.16.20 has a second moment of area, I, that varies such that $I = I_0 \sec\theta$ where I_0 is the second moment of area at the crown of the arch and θ is the slope of the tangent at any point. Calculate the horizontal thrust at the arch supports and determine the bending moment in the arch at the loading points and at the crown.

Ans. $R_{A,H} = R_{B,H} = 169.8$ kN $M_D = 47.2$ kN m $M_C = -9.4$ kNm.

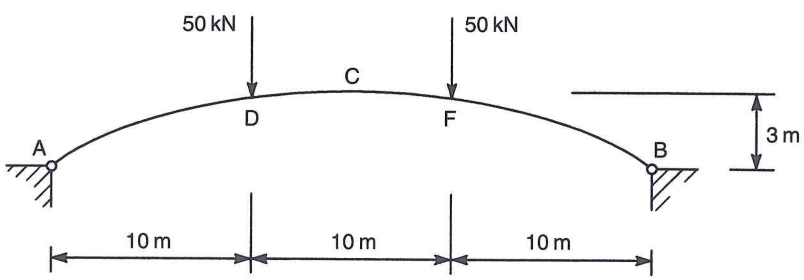


FIGURE P.16.20

16.21 Show that, for a two-pinned parabolic arch carrying a uniformly distributed load over its complete span and in which the second moment of area of the cross section varies as the secant assumption, the bending moment is everywhere zero.

16.22 The symmetrical two-pinned arch shown in Fig. P.16.22 comprises two segments each having a radius of 8 m. Calculate the horizontal reactions at the supports and the bending moment at the crown; the section properties are uniform throughout.

Ans. 20.4 kN, 58.7 kNm.

16.23 A two-pinned parabolic arch has its supports on the same horizontal level, a span of 40 m, a rise of 5 m and carries a vertical concentrated load of 10 kN acting at a point 10 m from the left-hand support. If the second moment of area of the arch cross section is governed by the secant rule calculate the horizontal thrust at the supports. With an origin of axes at the left-hand support the equation of the arch is

$y = 4h(Lx - x^2)/L^2$ where h is the rise of the arch and L its span. Calculate also the maximum bending moment in the arch.

Ans. 11.2 kN, 33.0 kNm.

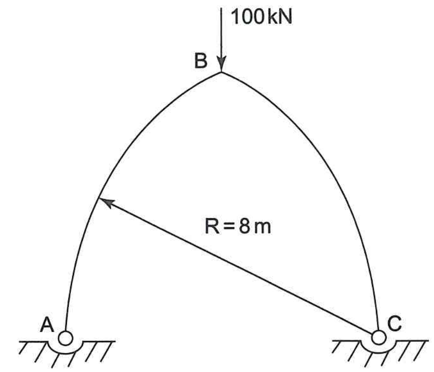


FIGURE P.16.22

P.16.24 A semi-circular two-pinned arch has a radius of 5 m and carries a uniformly distributed load of intensity 10 kN/m over its complete span. If the flexural rigidity EI of the arch is constant and the supports are on the same horizontal level calculate the support reactions.

Ans. 21.2 kN (horizontal), 50.0 kN (vertical).

P.16.25 The arch shown in Fig. P.16.25 is parabolic, the equation of its profile being $y = 0.05x(40 - x)$. If the second moment of area of the cross section of the arch is governed by the secant rule calculate the reactions at the supports and the bending moment at the crown.

Ans. $R_{A,H} = R_{B,H} = 3.15$ kN. $R_{A,V} = 11.6$ kN, $R_{B,V} = 3.4$ kN, $M_C = 19$ kNm.

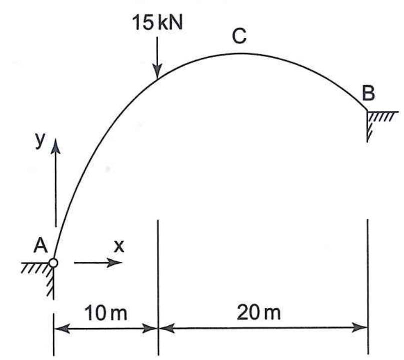


FIGURE P.16.25

P.16.26 A two-pinned semi-circular arch of constant cross section and radius R carries a central load W and has a flexural rigidity EI. If the coefficient of linear expansion of the material of the arch is α find the temperature change for there to be no horizontal reaction at the supports.

Ans. $-WR^2/4EI\alpha$.

P.16.27 Use the slope–deflection method to solve P.16.3 and P.16.4.

P.16.28 Use the slope–deflection method to determine the member end moments in the portal frame of Ex. 16.22.

P.16.29 Use the slope-deflection method to calculate the end moments in the beam shown in Fig. P.16.29; the flexural rigidity of the beam is EI. If the beam is now subjected to a downward displacement of 5 mm at the support B calculate the additional end moments this will produce.

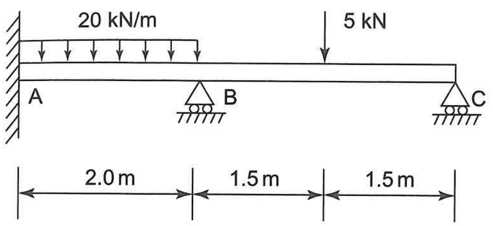


FIGURE P.16.29

P.16.30 Calculate the support reactions in the continuous beam shown in Fig. P.16.30 using the moment distribution method; the flexural rigidity, *EI*, of the beam is constant throughout.

Ans. $R_A = 2.7$ kN $R_B = 10.6$ kN $R_C = 3.7$ kN $M_A = -1.7$ kNm.

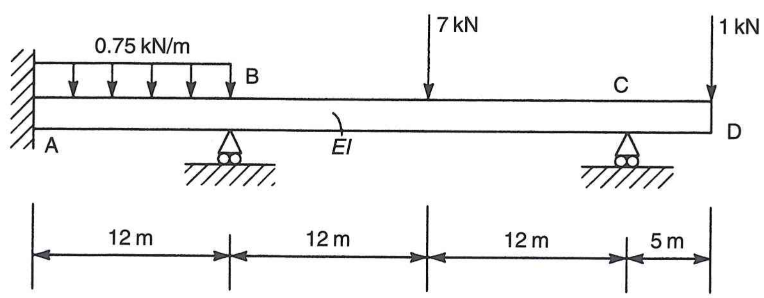


FIGURE P.16.30

P.16.31 Calculate the support reactions in the beam shown in Fig. P.16.31 using the moment distribution method; the flexural rigidity, *EI*, of the beam is constant throughout.

Ans. $R_C = 28.2$ kN $R_D = 17.0$ kN $R_E = 4.8$ kN $M_E = 1.6$ kNm.

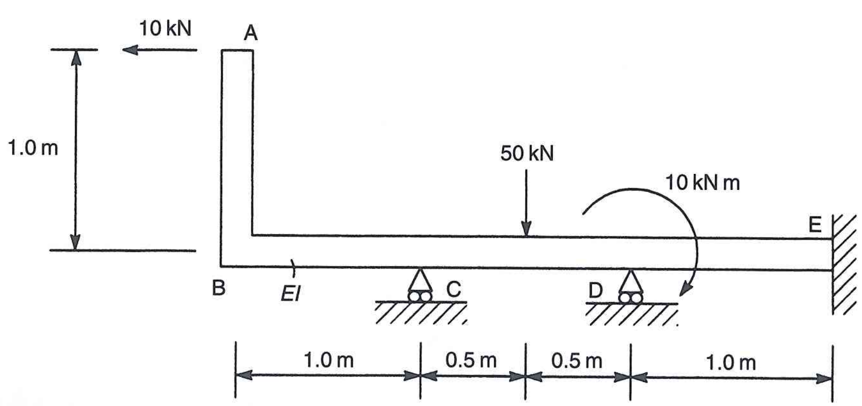


FIGURE P.16.31

P.16.32 In the beam ABC shown in Fig. P.16.32 the support at B settles by 10 mm when the loads are applied. If the second moment of area of the spans AB and BC are 83.4×10^6 mm^4 and 125.1×10^6 mm^4, respectively, and Young's modulus, *E*,of the material of the beam is 207000 N/mm^2, calculate the support reactions using the moment distribution method.

Ans. $R_C = 28.6$ kN $R_B = 15.8$ kN $R_A = 30.5$ kN $M_A = 53.9$ kNm.

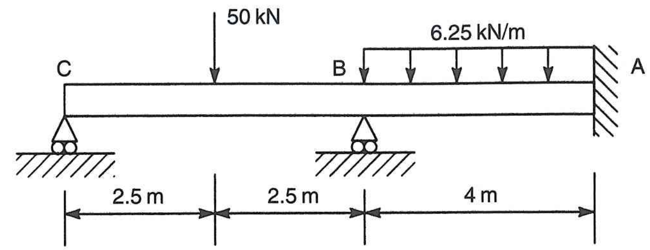


FIGURE P.16.32

P.16.33 Calculate the end moments in the members of the frame shown in Fig. P.16.33 using the moment distribution method. The flexural rigidity of the members AB, BC and BD are 2*EI*,3*EI* and *EI*, respectively, and the support system is such that sway is prevented.

Ans. $M_{AB} = M_{CB} = 0$ $M_{BA} = 30$ kNm $M_{BC} = -36$ kNm, $M_{BD} = 6$ kNm $M_{DB} = 3$ kNm.

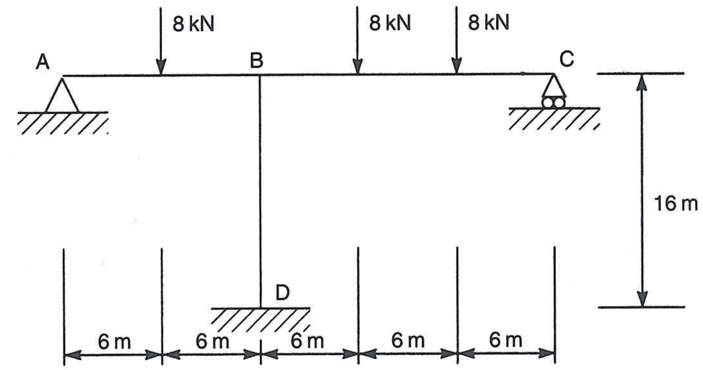
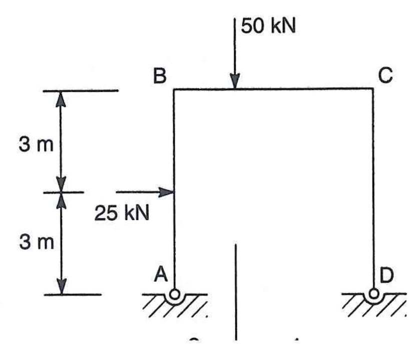


FIGURE P.16.33

P.16.34 The frame shown in Fig. P.16.34 is pinned to the foundation at A and D and has members whose flexural rigidity is *EI*. Use the moment distribution method to calculate the moments in the members and draw the bending moment diagram.

Ans. $M_A = M_D = 0$ $M_B = 11.9$ kNm $M_C = 63.2$ kNm.



P.16.35 Use the moment distribution method to calculate the bending moments at the joints in the frame shown in Fig. P.16.35 and draw the bending moment diagram.

Ans. $M_{AB} = M_{DC} = 0$ $M_{BA} = 12.7$ kNm $= -M_{BC}$ $M_{CB} = -13.9$ kNm $= -M_{CD}$.

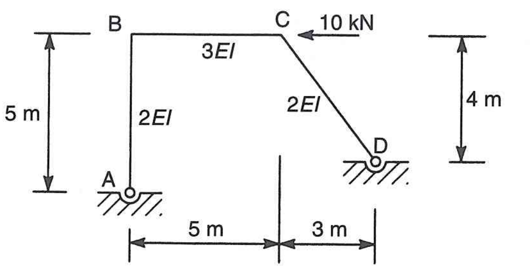


FIGURE P.16.35

P.16.36 The frame shown in Fig. P.16.36 has rigid joints at B, C and D and is pinned to its foundation at A and G. The joint D is prevented from moving horizontally by the member DF which is pinned to a support at F. The flexural rigidity of the members AB and BC is $2EI$ while that of all other members is EI.

Use the moment distribution method to calculate the end moments in the members.

Ans. $M_{BA} = -M_{BC} = 2.6$ kNm $M_{CB} = -M_{CD} = 67.7$ kNm $M_{DC} = -53.5$ kNm
$M_{DF} = 26.7$ kNm $M_{DG} = 26.7$ kNm.

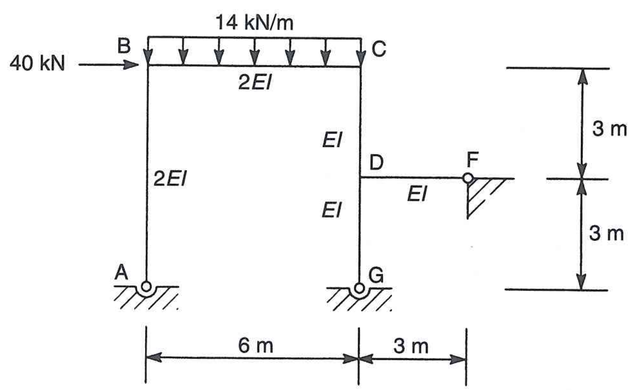


FIGURE P.16.36

P.16.37 The portal frame shown in Fig. P.16.37 is rigidly built in to its foundations at A and D while the joints B and C are rigid. Use the moment distribution method to calculate the bending moments at A, B, C and D and sketch the bending moment diagram.

Ans. A = 26.22 kNm, B = 7.63 kNm, C = 50.7 kNm, D = 41.66 kNm.

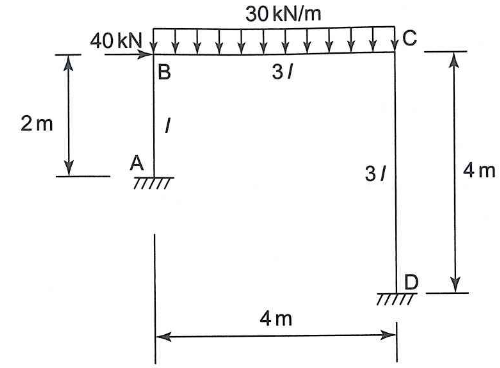


FIGURE P.16.37

P.16.38 The frame shown in Fig. P.16.38 is pinned to the foundation at A and D and has rigid joints at B and C. Use the moment distribution method to calculate the bending moments at B and C and sketch the bending moment diagram.

Ans. B = 16.5 kNm, C = 102.4 kNm.

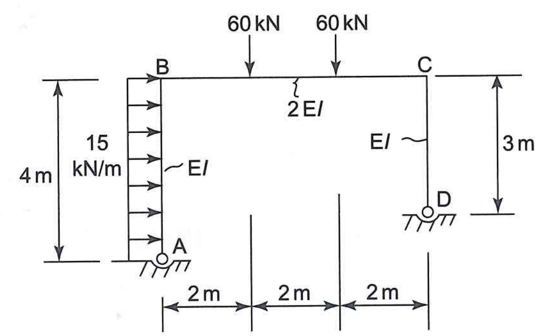


FIGURE P.16.38

P.16.39 The frame shown in Fig. P.16.39 is pinned to supports at A and E. Sidesway of the frame is prevented by a cross-brace BD which may be assumed to have zero bending stiffness but infinite axial stiffness. Use the moment distribution method to calculate the bending moments at the joints, the force in the member BD and the vertical and horizontal reactions at A and E. Also sketch the bending moment diagram for the frame and its displaced shape.

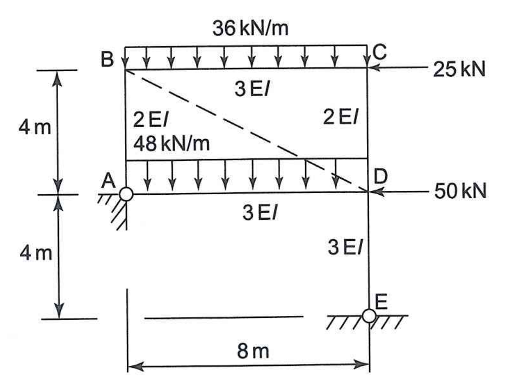


FIGURE P.16.39

Ans. $M_B = 115$ kNm, $M_C = 170$ kNm, M_D (in DC) = 178 kNm,
M_D (in DA) = 316 kNm, M_D (in DE) = 138 kNm.
Force in BD = 93.1 kN.
$R_{A,H} = 109.5$ kN to right, $R_{A,V} = 331.2$ kN upwards
$R_{E,H} = 34.5$ kN to left, $R_{E,V} = 340.8$ kN upwards.

P.16.40 A steel-framed extension is to be attached to an existing building by a pinned support D as shown in Fig. P.16.40; the frame is also pinned to foundations at E and F. Use the moment distribution method to determine the bending moments at the joints of the frame produced by the idealised wind loading shown. The relative second moments of area of the members of the frame are as shown. Sketch the bending moment diagram for the frame and its displaced shape.

Ans. $M_A = 15.5$ kNm, M_B (in BA) = 2.5 kNm, M_B (in BF) = 41.8 kNm,
M_B (in BC) = 39.4 kNm, $M_C = 23.3$ kNm.

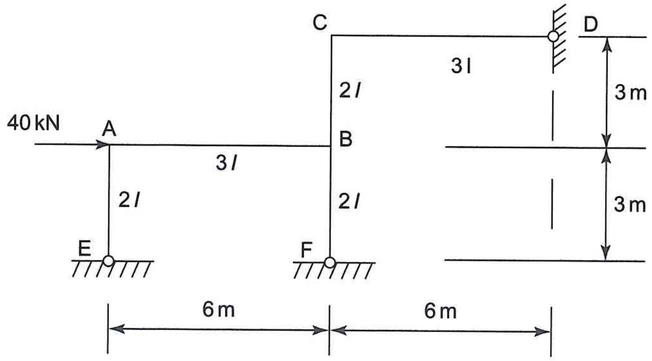


FIGURE P.16.40

CHAPTER

Matrix Methods of Analysis

17

The methods described in Chapter 16 are basically methods of analysis which are suitable for use with a hand calculator. They also provide an insight into the physical behaviour of structures under different loading conditions and it is this fundamental knowledge which enables the structural engineer to design structures which are capable of fulfilling their required purpose. However, the more complex a structure the lengthier, and more tedious, hand methods of analysis become and the more the approximations which have to be made. It was this situation which led, in the late 1940s and early 1950s, to the development of *matrix methods* of analysis and, at the same time, to the emergence of high-speed, electronic, digital computers. Conveniently, matrix methods are ideally suited to expressing structural theory in a form suitable for numerical solution by computer.

The modern digital computer is capable of storing vast amounts of data and producing solutions for highly complex structural problems almost instantaneously. There is a wide range of program packages available which cover static and dynamic problems in all types of structure from skeletal to continuum. Unfortunately these packages are not foolproof and so it is essential for the structural engineer to be able to select the appropriate package and to check the validity of the results; without a knowledge of fundamental theory this is impossible.

In Section 16.1 we discussed the flexibility and stiffness methods of analysis of statically indeterminate structures and saw that the flexibility method involves releasing the structure, determining the displacements in the released structure and then finding the forces required to fulfil the compatibility of displacement condition in the complete structure. The method was applied to statically indeterminate beams, trusses, braced beams, portal frames and two-pinned arches in Sections 16.4–16.8. It is clear from the analysis of these types of structure that the greater the degree of indeterminacy the higher the number of simultaneous equations requiring solution; for large numbers of equations a computer approach then becomes necessary. Furthermore, the flexibility method requires judgements to be made in terms of the release selected, so that a more automatic procedure is desirable so long, of course, as the fundamental behaviour of the structure is understood.

In Section 16.9 we examined the slope–deflection method for the solution of statically indeterminate beams and frames; the slope–deflection equations also form the basis of the moment–distribution method described in Section 16.10. These equations are, in fact, force–displacement relationships as opposed to the displacement–force relationships of the flexibility method. The slope–deflection and moment–distribution methods are therefore *stiffness* or *displacement* methods.

The stiffness method basically requires that a structure, which has a degree of *kinematic indeterminacy* equal to n_k, is initially rendered determinate by imposing a system of n_k constraints. Thus, for example, in the slope–deflection analysis of a continuous beam (e.g. Ex. 16.15) the beam is initially fixed at each support and the fixed-end moments calculated. This generally gives rise to an unbalanced system of forces at each node. Then by allowing displacements to occur at each node we obtain a series of force–displacement states (Eqs (i)–(vi) in Ex. 16.15). The n_k equilibrium conditions at the nodes are then expressed in terms of the displacements, giving n_k equations (Eqs (vii)–(x) in Ex. 16.15), the solution of which gives the true values of the displacements at the nodes. The internal stress resultants follow from the known force–displacement relationships for each member of the structure (Eqs (i)–(vi) in Ex. 16.15) and the complete solution is then the sum of the determinate solution and the set of n_k indeterminate systems.

Again, as in the flexibility method, we see that the greater the degree of indeterminacy (kinematic in this case) the greater the number of equations requiring solution, so that a computer-based approach is necessary when the degree of indeterminacy is high. Generally this requires that the force–displacement relationships in a structure are expressed in matrix form. We therefore need to establish force–displacement relationships for structural members and to examine the way in which these individual force–displacement relationships are combined to produce a force–displacement relationship for the complete structure. Initially we shall investigate members that are subjected to axial force only.

17.1 Axially loaded members

Consider the axially loaded member, AB, shown in Fig. 17.1(a) and suppose that it is subjected to axial forces, F_A and F_B, and that the corresponding displacements are w_A and w_B; the member has a cross-sectional area, A, and Young's modulus, E. An elemental length, δx, of the member is subjected to forces and displacements as shown in Fig. 17.1(b) so that its change in length from its unloaded state is $w + \delta w - w = \delta w$. Thus, from Eq. (7.4), the strain, ε, in the element is given by

$$\varepsilon = \frac{\mathrm{d}w}{\mathrm{d}x}$$

Further, from Eq. (7.8)

$$\frac{F}{A} = E\frac{\mathrm{d}w}{\mathrm{d}x}$$

so that

$$\mathrm{d}w = \frac{F}{AE}\mathrm{d}x$$

Therefore the axial displacement at the section a distance x from A is given by

$$w = \int_0^x \frac{F}{AE}\mathrm{d}x$$

which gives

$$w \frac{F}{AE}x + C_1$$

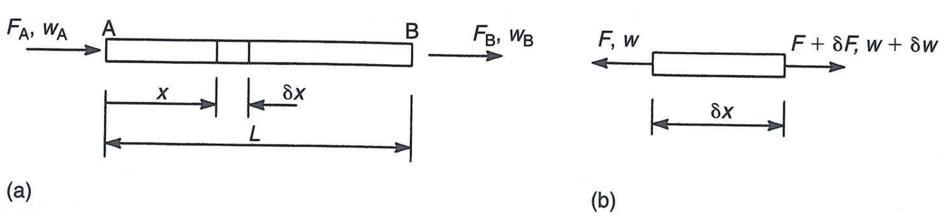


FIGURE 17.1

Axially loaded member

in which C_1 is a constant of integration. When $x = 0$, $w = w_A$ so that $C_1 = w_A$ and the expression for w may be written as

$$w_B = \frac{F}{AE}x + w_A \tag{17.1}$$

In the absence of any loads applied between A and B, $F = F_B = -F_A$ and Eq. (17.1) may be written as

$$w = \frac{F_B}{AE}x + w_A \tag{17.2}$$

Thus, when $x = L$, $w = w_B$ so that from Eq. (17.2)

$$w_B = \frac{F_B}{AE}L + w_A$$

or

$$F_B = \frac{AE}{L}(w_B - w_A) \tag{17.3}$$

Furthermore, since $F_B = -F_A$ we have, from Eq. (17.3)

$$-F_A = \frac{AE}{L}(w_B - w_A)$$

or

$$F_A = -\frac{AE}{L}(w_B - w_A) \tag{17.4}$$

Equations (17.3) and (17.4) may be expressed in matrix form as follows

$$\begin{Bmatrix} F_A \\ F_B \end{Bmatrix} = \begin{bmatrix} AE/L & -AE/L \\ -AE/L & AE/L \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \end{Bmatrix}$$

or

$$\begin{Bmatrix} F_A \\ F_B \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \end{Bmatrix} \tag{17.5}$$

Equation (17.5) may be written in the general form

$$\{F\} = [K_{AB}]\{w\} \tag{17.6}$$

in which $\{F\}$ and $\{w\}$ are generalized force and displacement matrices and $[K_{AB}]$ is the *stiffness matrix* of the member AB.

Suppose now that we have two axially loaded members, AB and BC, in line and connected at their common node B as shown in Fig. 17.2.

In Fig. 17.2 the force, F_B, comprises two components: $F_{B,AB}$ due to the change in length of AB, and $F_{B,BC}$ due to the change in length of BC. Thus, using the results of Eqs (17.3) and (17.4)

$$F_A = \frac{A_{AB}E_{AB}}{L_{AB}}(w_A - w_B) \tag{17.7}$$

$$F_B = F_{B,AB} + F_{B,BC} = \frac{A_{AB}E_{AB}}{L_{AB}}(w_B - w_A) + \frac{A_{BC}E_{BC}}{L_{BC}}(w_B - w_C) \tag{17.8}$$

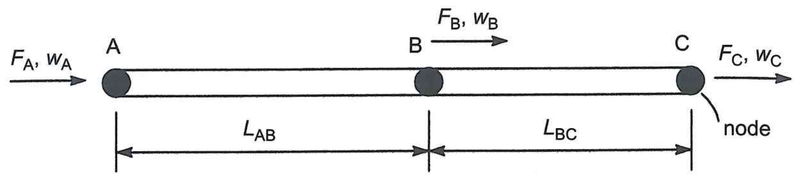


FIGURE 17.2

Two axially loaded members in line.

$$F_C = \frac{A_{BC}E_{BC}}{L_{BC}}(w_C - w_B) \tag{17.9}$$

in which A_{AB}, E_{AB} and L_{AB} are the cross-sectional area, Young's modulus and length of the member AB; similarly for the member BC. The term AE/L is a measure of the stiffness of a member, this we shall designate by k. Thus, Eqs (17.7)–(17.9) become

$$F_A = k_{AB}(w_A - w_B) \tag{17.10}$$

$$F_B = -k_{AB}w_A + (k_{AB} + k_{BC})w_B - k_{BC}w_C \tag{17.11}$$

$$F_C = k_{BC}(w_C - w_B) \tag{17.12}$$

Equations (17.10)–(17.12) are expressed in matrix form as

$$\begin{Bmatrix} F_A \\ F_B \\ F_C \end{Bmatrix} = \begin{bmatrix} k_{AB} & -k_{AB} & 0 \\ -k_{AB} & k_{AB}+k_{BC} & -k_{BC} \\ 0 & -k_{BC} & k_{BC} \end{bmatrix} \begin{Bmatrix} w_A \\ w_B \\ w_C \end{Bmatrix} \tag{17.13}$$

Note that in Eq. (17.13) the stiffness matrix is a symmetric matrix of order 3×3, which, as can be seen, connects *three* nodal forces to *three* nodal displacements. Also, in Eq. (17.5), the stiffness matrix is a 2×2 matrix connecting *two* nodal forces to *two* nodal displacements. We deduce, therefore, that a stiffness matrix for a structure in which n nodal forces relate to n nodal displacements will be a symmetric matrix of the order $n \times n$.

In more general terms the matrix in Eq. (17.13) may be written in the form

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{17.14}$$

in which the element k_{11} relates the force at node 1 to the displacement at node 1, k_{12} relates the force at node 1 to the displacement at node 2, and so on. Now, for the member connecting nodes 1 and 2

$$[k_{12}] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

and for the member connecting nodes 2 and 3

$$[k_{23}] = \begin{bmatrix} k_{22} & k_{23} \\ k_{32} & k_{33} \end{bmatrix}$$

Therefore we may assemble a stiffness matrix for a complete structure, not by the procedure used in

inserting them into the overall stiffness matrix such as that in Eq. (17.14). The element k_{22} appears in both $[K_{12}]$ and $[K_{23}]$ and will therefore receive contributions from both matrices. Hence, from Eq. (17.5)

$$[K_{AB}] = \begin{bmatrix} k_{AB} & -k_{AB} \\ -k_{AB} & k_{AB} \end{bmatrix}$$

and

$$[K_{BC}] = \begin{bmatrix} k_{BC} & -k_{BC} \\ -k_{BC} & k_{BC} \end{bmatrix}$$

Inserting these matrices into Eq. (17.14) we obtain

$$[K_{ABC}] = \begin{bmatrix} k_{AB} & -k_{AB} & 0 \\ -k_{AB} & k_{AB}+k_{BC} & -k_{BC} \\ 0 & -k_{BC} & k_{BC} \end{bmatrix}$$

as before. We see that only the k_{22} term (linking the force at node 2(B) to the displacement at node 2) receives contributions from both members AB and BC. This results from the fact that node 2(B) is directly connected to both nodes 1(A) and 3(C) while nodes 1 and 3 are connected directly to node 2. Nodes 1 and 3 are not directly connected so that the terms k_{13} and k_{31} are both zero, i.e. they are not affected by each other's displacement.

To summarize, the formation of the stiffness matrix for a complete structure is carried out as follows: terms of the form k_{ii} on the main diagonal consist of the sum of the stiffnesses of all the structural elements meeting at node i, while the off-diagonal terms of the form k_{ij} consist of the sum of the stiffnesses of all the elements connecting node i to node j.

Equation (17.13) may be solved for a specific case in which certain boundary conditions are specified. Thus, for example, the member AB may be fixed at A and loads F_B and F_C applied. Then $w_A = 0$ and F_A is a reaction force. Inversion of the resulting matrix enables w_B and w_C to be found.

In a practical situation a member subjected to an axial load could be part of a truss which would comprise several members set at various angles to one another. Therefore, to assemble a stiffness matrix for a complete structure, we need to refer axial forces and displacements to a common, or *global*, axis system.

Consider the member shown in Fig. 17.3. It is inclined at an angle θ to a global axis system denoted by xy. The member connects node i to node j, and has *member* or *local* axes $\bar{x}$, $\bar{y}$. Thus nodal forces and displacements referred to local axes are written as $\bar{F}$, $\bar{w}$, etc., so that, by comparison with Eq. (17.5), we see that

$$\begin{Bmatrix} \bar{F}_{x,i} \\ \bar{F}_{x,j} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \bar{w}_i \\ \bar{w}_j \end{Bmatrix} \tag{17.15}$$

where the member stiffness matrix is written as $[\bar{K}_{ij}]$.

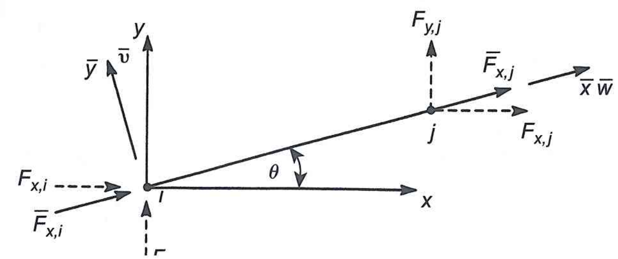


FIGURE 17.3

Local and global axes systems for an axially

In Fig. 17.3 external forces $\overline{F}_{x,i}$ and $\overline{F}_{x,j}$ are applied to i and j. It should be noted that $\overline{F}_{y,i}$ and $\overline{F}_{y,j}$ do not exist since the member can only support axial forces. However, $\overline{F}_{x,i}$ and $\overline{F}_{x,j}$ have components $F_{x,i}$, $F_{y,i}$ and $F_{x,j}$, $F_{y,j}$ respectively, so that whereas only two force components appear for the member in local coordinates, four components are present when global coordinates are used. Therefore, if we are to transfer from local to global coordinates, Eq. (17.15) must be expanded to an order consistent with the use of global coordinates. Thus

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \overline{w}_i \\ \overline{v}_i \\ \overline{w}_j \\ \overline{v}_j \end{Bmatrix} \tag{17.16}$$

Expansion of Eq. (17.16) shows that the basic relationship between $\overline{F}_{x,i}$, $\overline{F}_{x,j}$ and $\overline{w}_i$, $\overline{w}_j$ as defined in Eq. (17.15) is unchanged.

From Fig. 17.3 we see that

$$\overline{F}_{x,i} = F_{x,i}\cos\theta + F_{y,i}\sin\theta$$
$$\overline{F}_{y,i} = -F_{x,i}\sin\theta + F_{y,i}\cos\theta$$

and

$$\overline{F}_{x,j} = F_{x,j}\cos\theta + F_{y,j}\sin\theta$$
$$\overline{F}_{y,j} = -F_{x,j}\sin\theta + F_{y,j}\cos\theta$$

Writing λ for $\cos\theta$ and μ for $\sin\theta$ we express the above equations in matrix form as

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \end{Bmatrix} = \begin{bmatrix} \lambda & \mu & 0 & 0 \\ -\mu & \lambda & 0 & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -\mu & \lambda \end{bmatrix} \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{x,j} \\ F_{y,j} \end{Bmatrix} \tag{17.17}$$

or, in abbreviated form

$$\{\overline{F}\} = [T]\{F\} \tag{17.18}$$

where $[T]$ is known as the *transformation matrix.* A similar relationship exists between the sets of nodal displacements. Thus

$$\{\overline{\delta}\} = [T]\{\delta\} \tag{17.19}$$

in which $\{\overline{\delta}\}$ and $\{\delta\}$ are generalized displacements referred to the local and global axes, respectively. Substituting now for $\{\overline{F}\}$ and $\{\overline{\delta}\}$ in Eq. (17.16) from Eqs (17.18) and (17.19) we have

$$[T]\{F\} = [\overline{K}_{ij}][T]\{\delta\}$$

Hence

$$\{F\} = [T^{-1}][\overline{K}_{ij}][T]\{\delta\} \tag{17.20}$$

It may be shown that the inverse of the transformation matrix is its transpose, i.e.

$$[T^{-1}] = [T]^{\mathrm{T}}$$

Thus we rewrite Eq. (17.20) as

$$\{F\} = [T]^{\mathrm{T}}[\overline{K}_{ij}][T]\{\delta\} \tag{17.21}$$

The nodal force system referred to the global axes, $\{F\}$, is related to the corresponding nodal displacements by

$$\{F\} = [K_{ij}]\{\delta\} \tag{17.22}$$

in which $[K_{ij}]$ is the member stiffness matrix referred to global coordinates. Comparison of Eqs (17.21) and (17.22) shows that

$$\{K_{ij}\} = [T]^{\mathrm{T}}[\overline{K}_{ij}][T]$$

Substituting for $[T]$ from Eq. (17.17) and $[\overline{K}_{ij}]$ from Eq. (17.16) we obtain

$$[K_{ij}] = \frac{AE}{L} \begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & \mu^2 & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \tag{17.23}$$

Evaluating λ ($=\cos\theta$) and μ ($=\sin\theta$) for each member and substituting in Eq. (17.23) we obtain the stiffness matrix, referred to global axes, for each member of the framework.

EXAMPLE 17.1

Determine the horizontal and vertical components of the deflection of node 2 and the forces in the members of the truss shown in Fig. 17.4. The product AE is constant for all members.

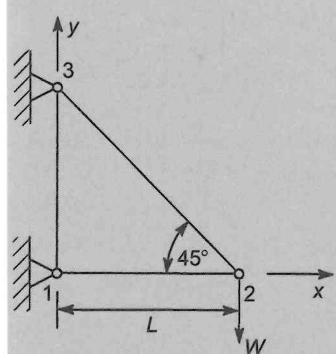


FIGURE 17.4

Truss of Ex. 17.1.

We see from Fig. 17.4 that the nodes 1 and 3 are pinned to the foundation and are therefore not displaced. Hence, referring to the global coordinate system shown,

$$w_1 = v_1 = w_3 = v_3 = 0$$

The external forces are applied at node 2 such that $F_{x,2} = 0$, $F_{y,2} = -W$; the nodal forces at 1 and 3 are then unknown reactions.

The first step in the solution is to assemble the stiffness matrix for the complete framework by writing down the member stiffness matrices referred to the global axes using Eq. (17.23). The direction cosines λ and μ take different values for each of the three members; therefore, remembering that the angle θ is measured anticlockwise from the positive direction of the x axis we have the following:

Member	θ (deg)	λ	μ
12	0	1	0
13	90	0	1
23	135	−0.707	0.707

The member stiffness matrices are therefore

$$[K_{12}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad [K_{13}] = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$[K_{23}] = \frac{AE}{1.414L}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \qquad \text{(i)}$$

The complete stiffness matrix is now assembled using the method suggested in the discussion of Eq. (17.14). The matrix will be a 6 × 6 matrix since there are six nodal forces connected to six nodal displacements; thus

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,2} \\ F_{y,2} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.354 & -0.354 & -0.354 & 0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & -1 & 0.354 & -0.354 & -0.354 & 1.354 \end{bmatrix}\begin{Bmatrix} w_1 = 0 \\ v_1 = 0 \\ w_2 \\ v_2 \\ w_3 = 0 \\ v_3 = 0 \end{Bmatrix} \qquad \text{(ii)}$$

If we now delete rows and columns in the stiffness matrix corresponding to zero displacements, we obtain the unknown nodal displacements w_2 and v_2 in terms of the applied loads $F_{x,2}$ (=0) and $F_{y,2}$ (=−W). Thus

$$\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1.354 & -0.354 \\ -0.354 & 0.354 \end{bmatrix}\begin{Bmatrix} w_2 \\ v_2 \end{Bmatrix} \qquad \text{(iii)}$$

Inverting Eq. (iii) gives

$$\begin{Bmatrix} w_2 \\ v_2 \end{Bmatrix} = \frac{L}{AE}\begin{bmatrix} 1 & 1 \\ 1 & 3.828 \end{bmatrix}\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} \qquad \text{(iv)}$$

from which

$$w_2 = \frac{L}{AE}(F_{x,2} + F_{y,2}) = \frac{-WL}{AE}$$

$$v_2 = \frac{L}{AE}(F_{x,2} + 3.828F_{y,2}) = \frac{-3.828WL}{AE}$$

The reactions at nodes 1 and 3 are now obtained by substituting for w_2 and v_2 from Eq. (iv) into Eq. (ii). Hence

$$\begin{Bmatrix} F_{x,1} \\ F_{y,1} \\ F_{x,3} \\ F_{y,3} \end{Bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -0.354 & 0.354 \\ 0.354 & -0.354 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 3.828 \end{bmatrix}\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{Bmatrix} F_{x,2} \\ F_{y,2} \end{Bmatrix}$$

giving

$$F_{x,1} = -F_{x,2} - F_{y,2} = W$$
$$F_{y,1} = 0$$
$$F_{x,3} = F_{y,2} = -W$$
$$F_{y,3} = W$$

The internal forces in the members may be found from the axial displacements of the nodes. Thus, for a member ij, the internal force F_{ij} is given by

$$F_{ij} = \frac{AE}{L}(\overline{w}_j - \overline{w}_i) \qquad \text{(v)}$$

But

$$\overline{w}_j = \lambda w_j + \mu v_j$$
$$\overline{w}_i = \lambda w_i + \mu v_i$$

Hence

$$\overline{w}_j - \overline{w}_i = \lambda(w_j - w_i) + \mu(v_j - v_i)$$

Substituting in Eq. (v) and rewriting in matrix form,

$$F_{ij} = \frac{AE}{L}[\lambda \quad \mu]\begin{Bmatrix} w_j & w_i \\ v_j & v_i \end{Bmatrix} \qquad \text{(vi)}$$

Thus, for the members of the framework

$$F_{12} = \frac{AE}{L}[1 \quad 0]\begin{Bmatrix} \dfrac{-WL}{AE} - 0 \\ \dfrac{-3.828WL}{AE} - 0 \end{Bmatrix} = -W \text{ (compression)}$$

$$F_{13} = \frac{AE}{L}[0 \quad 1]\begin{Bmatrix} 0-0 \\ 0-0 \end{Bmatrix} = 0 \text{ (obvious from inspection)}$$

$$F_{23} = \frac{AE}{1.414L}[-0.707 \quad 0.707]\begin{Bmatrix} 0 + \dfrac{WL}{AE} \\ 0 + \dfrac{3.828WL}{AE} \end{Bmatrix} = 1.414W \text{ (tension)}$$

The matrix method of solution for the statically determinate truss of Ex. 17.1 is completely general and therefore applicable to any structural problem. We observe from the solution that the question of statical determinacy of the truss did not arise. Statically indeterminate trusses are therefore solved in an identical manner with the stiffness matrix for each redundant member being included in the complete stiffness matrix as described above. Clearly, the greater the number of members the greater the size of the stiffness matrix, so that a computer-based approach is essential.

The procedure for the matrix analysis of space trusses is similar to that for plane trusses. The main difference lies in the transformation of the member stiffness matrices from local to global coordinates since, as we see from Fig. 17.5, axial nodal forces $\overline{F}_{x,i}$ and $\overline{F}_{x,j}$ have each, now, three global components $F_{x,i}$, $F_{y,i}$, $F_{z,i}$ and $F_{x,j}$, $F_{y,j}$, $F_{z,j}$ respectively. The member stiffness matrix referred to global coordinates is therefore of the order 6×6 so that $[K_{ij}]$ of Eq. (17.15) must be expanded to the same order to allow for this. Hence

$$[\overline{K}_{ij}] = \frac{AE}{L}\begin{array}{c} \begin{matrix} \overline{w}_i & \overline{v}_i & \overline{u}_i & \overline{w}_j & \overline{v}_j & \overline{u}_j \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad (17.24)$$

In Fig. 17.5 the member ij is of length L, cross-sectional area A and modulus of elasticity E. Global and local coordinate systems are designated as for the two-dimensional case. Further, we suppose that

$$\theta_{x\bar{x}} = \text{angle between } x \text{ and } \bar{x}$$
$$\theta_{x\bar{y}} = \text{angle between } x \text{ and } \bar{y}$$
$$\vdots$$
$$\theta_{z\bar{y}} = \text{angle between } z \text{ and } \bar{y}$$
$$\vdots$$

Therefore, nodal forces referred to the two systems of axes are related as follows

$$\left.\begin{aligned} \overline{F}_x &= F_x \cos\theta_{x\bar{x}} + F_y \cos\theta_{x\bar{y}} + F_z \cos\theta_{x\bar{z}} \\ \overline{F}_y &= F_x \cos\theta_{y\bar{x}} + F_y \cos\theta_{y\bar{y}} + F_z \cos\theta_{y\bar{z}} \\ \overline{F}_z &= F_x \cos\theta_{z\bar{x}} + F_y \cos\theta_{z\bar{y}} + F_z \cos\theta_{z\bar{z}} \end{aligned}\right\} \qquad (17.25)$$

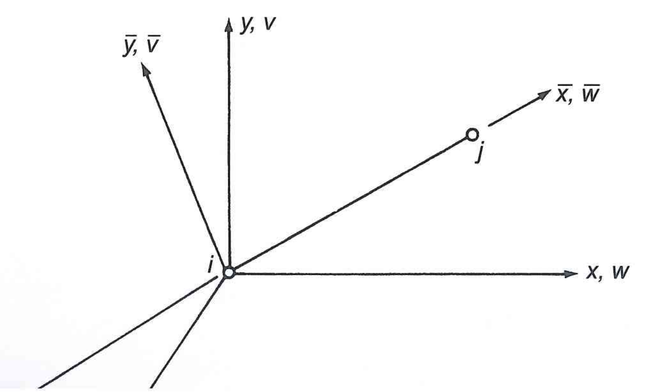


FIGURE 17.5

Writing

$$\left.\begin{matrix} \lambda_{\bar{x}} = \cos\theta_{x\bar{x}} & \lambda_{\bar{y}} = \cos\theta_{x\bar{y}} & \lambda_{\bar{z}} = \cos\theta_{x\bar{z}} \\ \mu_{\bar{x}} = \cos\theta_{y\bar{x}} & \mu_{\bar{y}} = \cos\theta_{y\bar{y}} & \mu_{\bar{z}} = \cos\theta_{y\bar{z}} \\ \nu_{\bar{x}} = \cos\theta_{z\bar{x}} & \nu_{\bar{y}} = \cos\theta_{z\bar{y}} & \nu_{\bar{z}} = \cos\theta_{z\bar{z}} \end{matrix}\right\} \qquad (17.26)$$

we may express Eq. (17.25) for nodes i and j in matrix form as

$$\begin{Bmatrix} \overline{F}_{x,i} \\ \overline{F}_{y,i} \\ \overline{F}_{z,i} \\ \overline{F}_{x,j} \\ \overline{F}_{y,j} \\ \overline{F}_{z,j} \end{Bmatrix} = \begin{bmatrix} \lambda_{\bar{x}} & \mu_{\bar{x}} & \nu_{\bar{x}} & 0 & 0 & 0 \\ \lambda_{\bar{y}} & \mu_{\bar{y}} & \nu_{\bar{y}} & 0 & 0 & 0 \\ \lambda_{\bar{z}} & \mu_{\bar{z}} & \nu_{\bar{z}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_{\bar{x}} & \mu_{\bar{x}} & \nu_{\bar{x}} \\ 0 & 0 & 0 & \lambda_{\bar{y}} & \mu_{\bar{y}} & \nu_{\bar{y}} \\ 0 & 0 & 0 & \lambda_{\bar{z}} & \mu_{\bar{z}} & \nu_{\bar{z}} \end{bmatrix} \begin{Bmatrix} F_{x,i} \\ F_{y,i} \\ F_{z,i} \\ F_{x,j} \\ F_{y,j} \\ F_{z,j} \end{Bmatrix} \qquad (17.27)$$

or in abbreviated form

$$\{\overline{F}\} = [T]\{F\}$$

The derivation of $[K_{ij}]$ for a member of a space frame proceeds on identical lines to that for the plane frame member. Thus, as before

$$[K_{ij}] = [T]^{\mathrm{T}}[\overline{K}_{ij}]\,[T]$$

Substituting for $[T]$ and $[\overline{K}_{ij}]$ from Eqs (17.27) and (17.24) gives

$$[K_{ij}] = \frac{AE}{L}\begin{bmatrix} \lambda_{\bar{x}}^2 & \lambda_{\bar{x}}\mu_{\bar{x}} & \lambda_{\bar{x}}\nu_{\bar{x}} & -\lambda_{\bar{x}}^2 & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\lambda_{\bar{x}}\nu_{\bar{x}} \\ \lambda_{\bar{x}}\mu_{\bar{x}} & \mu_{\bar{x}}^2 & \mu_{\bar{x}}\nu_{\bar{x}} & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\mu_{\bar{x}}^2 & -\mu_{\bar{x}}\nu_{\bar{x}} \\ \lambda_{\bar{x}}\nu_{\bar{x}} & \mu_{\bar{x}}\nu_{\bar{x}} & \nu_{\bar{x}}^2 & -\lambda_{\bar{x}}\nu_{\bar{x}} & -\mu_{\bar{x}}\nu_{\bar{x}} & -\nu_{\bar{x}}^2 \\ -\lambda_{\bar{x}}^2 & -\lambda_{\bar{x}}\mu_{\bar{x}} & -\lambda_{\bar{x}}\nu_{\bar{x}} & \lambda_{\bar{x}}^2 & \lambda_{\bar{x}}\mu_{\bar{x}} & \lambda_{\bar{x}}\nu_{\bar{x}} \\ -\lambda_{\bar{x}}\mu_{\bar{x}} & -\mu_{\bar{x}}^2 & -\mu_{\bar{x}}\nu_{\bar{x}} & \lambda_{\bar{x}}\mu_{\bar{x}} & \mu_{\bar{x}}^2 & \mu_{\bar{x}}\nu_{\bar{x}} \\ -\lambda_{\bar{x}}\nu_{\bar{x}} & -\mu_{\bar{x}}\nu_{\bar{x}} & -\nu_{\bar{x}}^2 & \lambda_{\bar{x}}\nu_{\bar{x}} & \mu_{\bar{x}}\nu_{\bar{x}} & \nu_{\bar{x}}^2 \end{bmatrix} \qquad (17.28)$$

All the suffixes in Eq. (17.28) are $\bar{x}$ so that we may rewrite the equation in simpler form, namely

$$[K_{ij}] = \frac{AE}{L}\left[\begin{array}{ccc:ccc} \lambda^2 & & & & & \textbf{SYM} \\ \lambda\mu & \mu^2 & & & & \\ \lambda\nu & \mu\nu & \nu^2 & & & \\ \hdashline -\lambda^2 & -\lambda\mu & -\lambda\nu & \lambda^2 & & \\ -\lambda\mu & -\mu^2 & -\mu\nu & \lambda\mu & \mu^2 & \\ -\lambda\nu & -\mu\nu & -\nu^2 & \lambda\nu & \mu\nu & \nu^2 \end{array}\right] \qquad (17.29)$$

where λ, μ and v are the direction cosines between the x, y, z and $\bar{x}$ axes, respectively.

The complete stiffness matrix for a space frame is assembled from the member stiffness matrices in a similar manner to that for the plane frame and the solution completed as before.

17.2 Stiffness matrix for a uniform beam

Our discussion so far has been restricted to structures comprising members capable of resisting axial

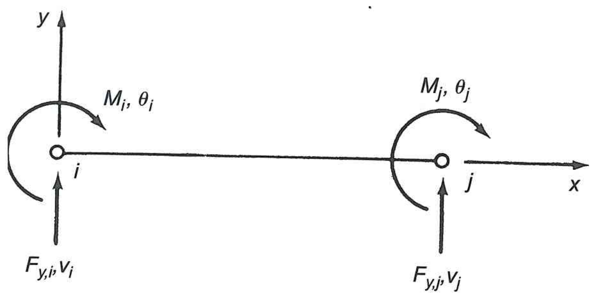


FIGURE 17.6

Forces and moments on a beam element.

esist shear and bending forces, in addition to axial loads. We shall now derive the stiffness matrix for a niform beam and consider the solution of rigid jointed frameworks formed by an assembly of beams, r beam elements as they are sometimes called.

Figure 17.6 shows a uniform beam ij of flexural rigidity EI and length L subjected to nodal forces $_{y,i}$, $F_{y,j}$ and nodal moments M_i, M_j in the xy plane. The beam suffers nodal displacements and rotaons v_i, v_j and θ_i, θ_j. We do not include axial forces here since their effects have already been deterined in our investigation of trusses.

The stiffness matrix $[K_{ij}]$ may be written down directly from the beam slope–deflection equations 6.27). Note that in Fig. 17.6 θi and θj are opposite in sign to θ_A and θ_B in Fig. 16.32. Then

$$M_i = -\frac{6EI}{L^2}v_i + \frac{4EI}{L}\theta_i + \frac{6EI}{L^2}v_j + \frac{4EI}{L}\theta_j \tag{17.28}$$

d

$$M_j = -\frac{6EI}{L^2}v_i + \frac{2EI}{L}\theta_i + \frac{6EI}{L^2}v_j + \frac{4EI}{L}\theta_j \tag{17.29}$$

Also

$$-F_{y,i} = F_{y,j} = -\frac{12EI}{L^3}v_i + \frac{6EI}{L^2}\theta_i + \frac{12EI}{L^3}v_j + \frac{6EI}{L^2}\theta_j \tag{17.30}$$

pressing Eqs (17.28), (17.29) and (17.30) in matrix form yields

$$\begin{Bmatrix} F_{y,i} \\ M_i \\ F_{y,j} \\ M_j \end{Bmatrix} = EI \begin{bmatrix} 12/L^3 & -6/L^2 & -12/L^3 & -6/L^2 \\ -6/L^2 & 4/L & 6/L^2 & 2/L \\ -12/L^3 & 6/L^2 & 12/L^3 & 6/L^2 \\ -6/L^2 & 2/L & 6/L^2 & 4/L \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \tag{17.31}$$

ich is of the form

$$\{F\} = [Kij]\{\delta\}$$

ere $[K_{ij}]$ is the stiffness matrix for the beam.

It is possible to write Eq. (17.31) in an alternative form such that the elements of $[K_{ij}]$ are pure nbers. Thus

$$\begin{Bmatrix} F_{y,i} \\ M_i/L \\ F_{y,j} \\ M_j/L \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & -6 & -12 & -6 \\ -6 & 4 & 6 & 2 \\ -12 & 6 & 12 & 6 \\ -6 & 2 & 6 & 4 \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i L \\ v_j \\ \theta_j L \end{Bmatrix}$$

This form of Eq. (17.31) is particularly useful in numerical calculations for an assemblage of beams in which EI/L^3 is constant.

Equation (17.31) is derived for a beam whose axis is aligned with the x axis so that the stiffness matrix defined by Eq. (17.31) is actually $[\overline{K}_{ij}]$ the stiffness matrix referred to a local coordinate system. If the beam is positioned in the xy plane with its axis arbitrarily inclined to the x axis then the x and y axes form a global coordinate system and it becomes necessary to transform Eq. (17.31) to allow for this. The procedure is similar to that for the truss member of Section 17.1 in that $[\overline{K}_{ij}]$ must be expanded to allow for the fact that nodal displacements $\overline{w}_i$ and $\overline{w}_j$, which are irrelevant for the beam in local coordinates, have components w_i, v_i and w_j, v_j in global coordinates. Thus

$$[\overline{K}_{ij}] = EI \begin{array}{c} \begin{matrix} w_i & v_i & \theta_i & w_j & v_j & \theta_j \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12/L^3 & -6/L^2 & 0 & -12/L^3 & -6/L^2 \\ 0 & -6/L^2 & 4/L & 0 & 6/L^2 & 2/L \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12/L^3 & 6/L^2 & 0 & 12/L^3 & 6/L^2 \\ 0 & -6/L^2 & 2/L & 0 & 6/L^2 & 4/L \end{bmatrix} \end{array} \tag{17.32}$$

We may deduce the transformation matrix $[T]$ from Eq. (17.17) if we remember that although w and v transform in exactly the same way as in the case of a truss member the rotations θ remain the same in either local or global coordinates.

Hence

$$[T] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17.33}$$

where λ and μ have previously been defined. Thus since

$$[K_{ij}] = [T]^{\mathrm{T}}[\overline{K}_{ij}][T]$$

we have, from Eqs (17.32) and (17.33)

$$[K_{ij}] = EI \begin{bmatrix} 12\mu^2/L^3 & & & & & \text{SYM} \\ -12\lambda\mu/L^3 & 12\lambda^2/L^3 & & & & \\ 6\mu/L^2 & -6\lambda/L^2 & 4/L & & & \\ -12\mu^2/L^3 & 12\lambda\mu/L^3 & -6\mu/L^2 & 12\mu^2/L^3 & & \\ 12\lambda\mu/L^3 & -12\lambda^2/L^3 & 6\lambda/L^2 & -12\lambda\mu/L^3 & 12\lambda^2/L^3 & \\ 6\mu/L^2 & -6\lambda/L^2 & 2/L & 6\mu/L^2 & 6\lambda/L^2 & 4\lambda/L \end{bmatrix} \tag{17.34}$$

Again the stiffness matrix for the complete structure is assembled from the member stiffness matrices, the boundary conditions are applied and the resulting set of equations solved for the unknown nodal displacements and forces.

The internal shear forces and bending moments in a beam may be obtained in terms of the calculated nodal displacements. Thus, for a beam joining nodes i and j we shall have obtained the unknown